

Backward Euler method for the Equations of Motion Arising in Oldroyd Fluids of Order One with Nonsmooth Initial Data

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Abstract

In this paper, a backward Euler method is discussed for the equations of motion arising in the 2D Oldroyd model of viscoelastic fluids of order one with the forcing term independent of time or in L^∞ in time. It is shown that the estimates of the discrete solution in Dirichlet norm is bounded uniformly in time. Optimal *a priori* error estimate in \mathbf{L}^2 -norm is derived for the discrete problem with non-smooth initial data. This estimate is shown to be uniform in time, under the assumption of uniqueness condition.

Key Words. Viscoelastic fluids, Oldroyd fluid of order one, backward Euler method, uniform in time bound, optimal and uniform error estimates, non-smooth initial data.

1 Introduction

In this paper, we consider fully-discrete approximations to the equations of motion arising in the Oldroyd fluids (see Oldroyd[15], Oskolkov[16]) of order one:

$$(1.1) \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \int_0^t \beta(t - \tau) \Delta \mathbf{u}(\tau) d\tau + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \quad t > 0$$

with incompressibility condition

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0, \quad \text{on } \Omega, \quad t > 0,$$

and initial and boundary conditions

$$(1.3) \quad \mathbf{u}(x, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \quad \mathbf{u} = 0, \quad \text{on } \partial\Omega, \quad t \geq 0.$$

Here, Ω is a bounded domain in \mathbb{R}^2 with boundary $\partial\Omega$, $\mu = 2\kappa\lambda^{-1} > 0$ and the kernel $\beta(t) = \gamma \exp(-\delta t)$, where $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$ and $\delta = \lambda^{-1} > 0$. Further, \mathbf{f} and \mathbf{u}_0 are given functions in their respective domain of definition. For more details, we refer to [1] and [15].

There is considerable amount of literature devoted to Oldroyd model by Oskolkov, Kotsiolis, Karzeeva, Sobolevskii etc, see [1, 5, 12, 13, 16] and recently by Lin *et al.* [9, 10, 24], Pani *et al.* [19, 20], Wang *et al.* [25], and references, therein. A detailed report on the continuous and semi-discrete cases can be found in [8].

Literature for the fully-discrete approximations to the problem (1.1)-(1.3) is, however, limited. In [2], Akhmatov and Oskolkov have discussed stable and convergent finite difference schemes for the problem (1.1)-(1.3). Recently in [20], a linearized backward Euler method is used to discretize in temporal direction and semi-group theoretic approach is then employed to establish *a priori* error estimates. The following error bounds are proved in [20] for $t_n > 0$

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq C e^{-\alpha t_n} k$$

and

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \leq C e^{-\alpha t_n} k (t_n^{-1/2} + \log \frac{1}{k})$$

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for smooth initial data and for zero forcing term. Here, k is the time step and \mathbf{U}^n is the finite difference approximation to $\mathbf{u}(t_n)$, when modified backward Euler method is applied in the temporal direction. Recently Wang *et al.* [25] have again applied backward Euler method for the problem (1.1)-(1.3), with smooth initial data, when the forcing function is non-zero. They have used energy arguments along with uniqueness condition to obtain the following uniform error estimates:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq C(h^2 + k)$$

and

$$\tau^{1/2} \|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \leq C(h + k),$$

where $\tau(t_n) = \min\{1, t_n\}$ and h is the mesh size, again with smooth initial data.

Our present investigation is a continuation of [8], where *a priori* estimates and regularity results have been established, which are uniform in time under realistically assumed regularity on the exact solution and when $\mathbf{f}, \mathbf{f}_t \in L^\infty(\mathbf{L}^2)$. Error estimates for semi-discrete Galerkin approximations have been shown to be optimal in $L^\infty(\mathbf{L}^2)$ -norm for non-smooth initial data. Further, uniform (in time) error estimates under uniqueness condition are also established.

In the present article, we discuss backward Euler method to discretize in the temporal variable and Galerkin approximations to discretize spatial variables for approximating solutions of the problem (1.1)-(1.3). Our main aim, in this work, is to present optimal error estimate for the backward Euler method, when the initial data is non-smooth, that is, $\mathbf{u}_0 \in \mathbf{J}_1$. The main results of this paper are follows:

- (i) Proving uniform bound in time in the Dirichlet norm for the solution of the completely discrete backward Euler method.
- (ii) Deriving new estimates which are valid uniformly in time for the error associated with discrete linearized problem
- (iii) Establishing estimates for the error related to nonlinear part in which the error constant depends exponentially in time and thereby, making final error estimate in the velocity to depend on exponentially in time.
- (iv) Proving optimal error estimates for the velocity in \mathbf{L}^2 -norm which are uniform in time under the uniqueness assumption.

To prove estimate in the Dirichlet norm for the discrete solution which is valid for all time, the usual tool, in the case of the Navier-Stokes equations, is to apply discrete version of uniform Gronwall's Lemma. Now for proving (i), it is difficult to apply uniform Gronwall's Lemma due to presence of the discrete version of integral term. Therefore, a new way of looking at the proof has helped to achieve (i), see; Lemma 4.3. For (ii) – (iii), we observe that there are difficulties due to the non-linear term along with the presence of integral term in the case of non-smooth initial data. For example, the preliminary result ($L^\infty(\mathbf{L}^2)$ estimate) is sub-optimal due to non-smooth initial data (see; Lemma 5.2). In order to compensate the loss in the order of convergence, a more appropriate tool is to multiply by t . But, unfortunately, it fails here due to the presence of the integral term (or the summation term). To overcome this difficulty, we modify some tools from the error analysis of linear parabolic integro-differential equations with non-smooth data (see; [17, 18, 23]) to fit into the present problem and also a special care is taken to bound the nonlinear term. Our analysis makes use of the solution, say; \mathbf{V}^n of a linearized discrete problem (see; (5.5)) as an intermediate solution. Then, with its help, we split the error: $\mathbf{u}_h^n - \mathbf{U}^n$ at time level $t = t_n$, where $\mathbf{u}_h^n = \mathbf{u}_h(t_n)$ is the solution of the semi-discrete scheme at $t = t_n$ and \mathbf{U}^n is the solution of the backward Euler method, into two error components: one in $\boldsymbol{\xi}^n := \mathbf{u}_h^n - \mathbf{V}^n$, which denotes the contribution due to the linearized part (see; (5.6)), and the other in $\boldsymbol{\eta}^n := \mathbf{U}^n - \mathbf{V}^n$, which is due to the non-linearity (see; (5.6)). Using a backward discrete linear problem and duality type argument along with an estimate of $\hat{\boldsymbol{\xi}}^n$, where

$$\hat{\boldsymbol{\xi}}^n := k \sum_{j=0}^n \boldsymbol{\xi}^j,$$

an L^2 -estimate of $\boldsymbol{\xi}^n$ which is valid for all time is derived, refer to Theorem 5.1. For L^2 estimate of $\boldsymbol{\eta}^n$, we employ negative norm estimate and L^2 estimate of $\hat{\boldsymbol{\eta}}^n$ and obtain estimate which depends on exponentially in time, see; Lemma 5.9. Thus, one of the main result for nonsmooth initial data that we have derived in Theorem 5.2 is as follows:

$$(1.4) \quad \|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq K_T t_n^{-1/2} (h^2 + k(1 + \log \frac{1}{k})^{1/2}),$$

where K_T depends exponentially on time. Finally for the proof of (iv), a careful use of the uniqueness condition, it is also shown that the error estimate (1.4) is valid for all time.

The remaining part of this paper is organized as follows. In Section 2, we discuss some notations, basic assumptions and weak formulations. In Section 3, a semidiscrete Galerkin method is discussed briefly. Section 4 is devoted to backward Euler method. Optimal and uniform error bounds are obtained for the velocity when the initial data are in \mathbf{J}_1 . Finally, we summarize our results in the Section 5.

2 Preliminaries

For our subsequent use, we denote by bold face letters the \mathbb{R}^2 -valued function space such as

$$\mathbf{H}_0^1 = [H_0^1(\Omega)]^2, \quad \mathbf{L}^2 = [L^2(\Omega)]^2 \quad \text{and} \quad \mathbf{H}^m = [H^m(\Omega)]^2,$$

where $H^m(\Omega)$ is the standard Hilbert Sobolev space of order m . Note that \mathbf{H}_0^1 is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^2 (\partial_j v_i, \partial_j v_i) \right)^{1/2} = \left(\sum_{i=1}^2 (\nabla v_i, \nabla v_i) \right)^{1/2}.$$

Further, we introduce some more function spaces for our future use:—

$$\begin{aligned} \mathbf{J}_1 &= \{ \phi \in \mathbf{H}_0^1 : \nabla \cdot \phi = 0 \} \\ \mathbf{J} &= \{ \phi \in \mathbf{L}^2 : \nabla \cdot \phi = 0 \text{ in } \Omega, \phi \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly} \}, \end{aligned}$$

where \mathbf{n} is the outward normal to the boundary $\partial\Omega$ and $\phi \cdot \mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [22]. Let H^m/\mathbb{R} be the quotient space consisting of equivalence classes of elements of H^m differing by constants, which is equipped with norm $\|p\|_{H^m/\mathbb{R}} = \|p + c\|_m$. For any Banach space X , let $L^p(0, T; X)$ denote the space of measurable X -valued functions ϕ on $(0, T)$ such that

$$\int_0^T \|\phi(t)\|_X^p dt < \infty \quad \text{if } 1 \leq p < \infty,$$

and for $p = \infty$

$$\operatorname{ess\,sup}_{0 < t < T} \|\phi(t)\|_X < \infty \quad \text{if } p = \infty.$$

Through out this paper, we make the following assumptions:

(A1). For $\mathbf{g} \in \mathbf{L}^2$, let the unique pair of solutions $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/R\}$ for the steady state Stokes problem

$$\begin{aligned} -\Delta \mathbf{v} + \nabla q &= \mathbf{g}, \\ \nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0, \end{aligned}$$

satisfy the following regularity result

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/R} \leq C \|\mathbf{g}\|.$$

(A2). The initial velocity \mathbf{u}_0 and the external force \mathbf{f} satisfy for positive constant M_0 , and for T with $0 < T \leq \infty$

$$\mathbf{u}_0 \in \mathbf{J}_1, \quad \mathbf{f}, \mathbf{f}_t \in L^\infty(0, T; \mathbf{L}^2) \quad \text{with} \quad \|\mathbf{u}_0\|_1 \leq M_0, \quad \sup_{0 < t < T} \{\|\mathbf{f}\|, \|\mathbf{f}_t\|\} \leq M_0.$$

For our subsequent analysis, we present the following Lemma, which can be seen as a discrete version of Lemma 2.2 from [19].

Lemma 2.1. *Let $g_i, \phi^i \in \mathbb{R}$, $1 \leq i \leq n$, $n \in \mathbb{N}$ and $0 < k < 1$. Then the following estimate holds*

$$\left(k \sum_{i=1}^n \left(k \sum_{j=1}^i g_{i-j} \phi^j \right)^2 \right)^{1/2} \leq \left(k \sum_{i=1}^k |g_i| \right) \left(k \sum_{i=1}^n |\phi^i|^2 \right)^{1/2}.$$

3 Semidiscrete Galerkin Approximations

From now on, we denote h with $0 < h < 1$ to be a real positive discretization parameter tending to zero. Let \mathbf{H}_h and L_h , $0 < h < 1$ be two family of finite dimensional subspaces of \mathbf{H}_0^1 and L^2 , respectively, approximating velocity vector and the pressure. Assume that the following approximation properties are satisfied for the spaces \mathbf{H}_h and L_h :

(B1) For each $\mathbf{w} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $q \in H^1/R$ there exist approximations $i_h \mathbf{w} \in \mathbf{H}_h$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h \mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h \mathbf{w})\| \leq K_0 h^2 \|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/R} \leq K_0 h \|q\|_{H^1/R}.$$

Further, suppose that the following inverse hypothesis holds for $\mathbf{w}_h \in \mathbf{H}_h$:

$$(3.1) \quad \|\nabla \mathbf{w}_h\| \leq K_0 h^{-1} \|\mathbf{w}_h\|.$$

For defining the Galerkin approximations, set for $\mathbf{v}, \mathbf{w}, \phi \in \mathbf{H}_0^1$,

$$a(\mathbf{v}, \phi) = (\nabla \mathbf{v}, \nabla \phi)$$

and

$$b(\mathbf{v}, \mathbf{w}, \phi) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \phi) - \frac{1}{2}(\mathbf{v} \cdot \nabla \phi, \mathbf{w}).$$

Note that the operator $b(\cdot, \cdot, \cdot)$ preserves the antisymmetric property of the original nonlinear term, that is,

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h.$$

Now, the semidiscrete Galerkin formulation reads as: Find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$(3.2) \quad \begin{aligned} (\mathbf{u}_{ht}, \phi_h) + \mu a(\mathbf{u}_h, \phi_h) &+ b(\mathbf{u}_h, \mathbf{u}_h, \phi_h) + a(\mathbf{u}_{h,\beta}, \phi_h) - (p_h, \nabla \cdot \phi_h) = (\mathbf{f}, \phi_h), \\ (\nabla \cdot \mathbf{u}_h, \chi_h) &= 0, \end{aligned}$$

for $\phi_h \in \mathbf{H}_h$, $\chi_h \in L_h$. Here $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$ and

$$(3.3) \quad \mathbf{u}_{h,\beta}(t) = \int_0^t \beta(t-s) \mathbf{u}_h(s) ds.$$

In order to consider a discrete space analogous to \mathbf{J}_1 , we impose the discrete incompressibility condition on \mathbf{H}_h and call it as \mathbf{J}_h . Thus, we define \mathbf{J}_h , as

$$\mathbf{J}_h = \{v_h \in \mathbf{H}_h : (\chi_h, \nabla \cdot v_h) = 0 \quad \forall \chi_h \in L_h\}.$$

Note that \mathbf{J}_h is not a subspace of \mathbf{J}_1 . With \mathbf{J}_h as above, we now introduce an equivalent Galerkin formulation as: Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$(3.4) \quad (\mathbf{u}_{ht}, \phi_h) + \mu a(\mathbf{u}_h, \phi_h) + a(\mathbf{u}_{h,\beta}, \phi_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \phi_h) + (\mathbf{f}, \phi_h) \quad \forall \phi_h \in \mathbf{J}_h.$$

Since \mathbf{J}_h is finite dimensional, the problem (3.4) leads to a system of nonlinear integro-differential equations. For global existence of a solution pair of (3.4), we refer to [19]. Uniqueness (of p) is obtained on the quotient space L_h/N_h , where

$$N_h = \{q_h \in L_h : (q_h, \nabla \cdot \phi_h) = 0, \forall \phi_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

For continuous dependence of the discrete pressure $p_h(t) \in L_h/N_h$ on the discrete velocity $u_h(t) \in \mathbf{J}_h$, we assume the following discrete inf-sup (LBB) condition for the finite dimensional spaces \mathbf{H}_h and L_h :

(B2') For every $q_h \in L_h$, there exists a non-trivial function $\phi_h \in \mathbf{H}_h$ and a positive constant K_0 , independent of h , such that

$$|(q_h, \nabla \cdot \phi_h)| \geq K_0 \|\nabla \phi_h\| \|q_h\|_{L^2/N_h}.$$

Moreover, we also assume that the following approximation property holds true for \mathbf{J}_h .

(B2) For every $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$, there exists an approximation $r_h \mathbf{w} \in \mathbf{J}_h$ such that

$$\|\mathbf{w} - r_h \mathbf{w}\| + h\|\nabla(\mathbf{w} - r_h \mathbf{w})\| \leq K_5 h^2 \|\mathbf{w}\|_2.$$

This is a less restrictive condition than (B2') and it has been used to derive the following properties of the L^2 projection $P_h : \mathbf{L}^2 \mapsto \mathbf{J}_h$. We now state without proof these results. For a proof, see [11]. For $\phi \in \mathbf{J}_h$, note that

$$(3.5) \quad \|\phi - P_h \phi\| + h\|\nabla P_h \phi\| \leq Ch\|\nabla \phi\|,$$

and for $\phi \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$(3.6) \quad \|\phi - P_h \phi\| + h\|\nabla(\phi - P_h \phi)\| \leq Ch^2 \|\tilde{\Delta} \phi\|.$$

We now define the discrete operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ through the bilinear form $a(\cdot, \cdot)$ as

$$(3.7) \quad a(\mathbf{v}_h, \phi_h) = (-\Delta_h \mathbf{v}_h, \phi) \quad \forall \mathbf{v}_h, \phi_h \in \mathbf{H}_h.$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h \Delta_h$. Using Sobolev imbedding and Sobolev inequality, it is easy to prove the following Lemma

Lemma 3.1. *Suppose conditions (A1), (B1) and (B2) are satisfied. Then there exists a positive constant K such that for $\mathbf{v}, \mathbf{w}, \phi \in \mathbf{H}_h$, the following holds:*

$$(3.8) \quad |(\mathbf{v} \cdot \nabla \mathbf{w}, \phi)| \leq K \begin{cases} \|\mathbf{v}\|^{1/2} \|\nabla \mathbf{v}\|^{1/2} \|\nabla \mathbf{w}\|^{1/2} \|\Delta_h \mathbf{w}\|^{1/2} \|\phi\|, \\ \|\mathbf{v}\|^{1/2} \|\Delta_h \mathbf{v}\|^{1/2} \|\nabla \mathbf{w}\| \|\phi\|, \\ \|\mathbf{v}\|^{1/2} \|\nabla \mathbf{v}\|^{1/2} \|\nabla \mathbf{w}\| \|\phi\|^{1/2} \|\nabla \phi\|^{1/2}, \\ \|\mathbf{v}\| \|\nabla \mathbf{w}\| \|\phi\|^{1/2} \|\Delta_h \phi\|^{1/2}, \\ \|\mathbf{v}\| \|\nabla \mathbf{w}\|^{1/2} \|\Delta_h \mathbf{w}\|^{1/2} \|\phi\|^{1/2} \|\nabla \phi\|^{1/2}. \end{cases}$$

Examples of subspaces \mathbf{H}_h and L_h satisfying assumptions (B1), (B2'), and (B2) can be found in [6, 4, 3].

We present below, a Lemma, that deals with higher order estimates of \mathbf{u}_h , which will be useful in the error analysis of backward Euler method for non-smooth data.

Lemma 3.2. *Suppose conditions (A1), (B1), (B2) and (B4) are satisfied. Moreover, let $\mathbf{u}_h(0) \in \mathbf{J}_h$ and \mathbf{f} satisfy the assumption (A3). Then, \mathbf{u}_h , the solutions of the semidiscrete Oldroyd problem (3.4) satisfies the following a priori estimates:*

$$(3.9) \quad \tau^* \|\mathbf{u}_h\|_2^2 + (\tau^*)^{r+1} \|\mathbf{u}_{ht}\|_r^2 \leq K, \quad r \in \{0, 1\},$$

$$(3.10) \quad e^{-2\alpha t} \int_0^t e^{2\alpha s} (\tau^*)^r(s) \|\mathbf{u}_{hs}\|_r^2 ds \leq K, \quad r \in \{0, 1, 2\},$$

$$(3.11) \quad e^{-2\alpha t} \int_0^t e^{2\alpha s} (\tau^*)^{r+1}(s) \|\mathbf{u}_{hss}\|_{r-1}^2 ds \leq K, \quad r \in \{-1, 0, 1\},$$

where $(\tau^*)(t) = \min\{1, t\}$, $\sigma(t) = \tau^*(t)e^{2\alpha t}$ and K depends on the given data, but not on time T .

Proof. The estimates (3.9)-(3.10) can be proved as in the continuous case, see [8]. For the final estimate, we differentiate (3.4) to find that, for $\phi_h \in \mathbf{J}_h$,

$$(3.12) \quad \begin{aligned} (\mathbf{u}_{htt}, \phi_h) + \mu a(\mathbf{u}_{ht}, \phi_h) &+ \beta(0)a(\mathbf{u}_h, \phi_h) - \delta \int_0^t \beta(t-s)a(\mathbf{u}_h(s), \phi_h) ds \\ &= -b(\mathbf{u}_{ht}, \mathbf{u}_h, \phi_h) - b(\mathbf{u}_h, \mathbf{u}_{ht}, \phi_h) + (\mathbf{f}_t, \phi_h). \end{aligned}$$

Taking $\phi_h = (\tau^*)^2(t)e^{2\alpha t} \mathbf{u}_{htt}$ in (3.12), we obtain

$$(3.13) \quad \begin{aligned} (\tau^*)^2(t)e^{2\alpha t} \|\mathbf{u}_{htt}\|^2 + \frac{\mu}{2} \frac{d}{dt} ((\tau^*)^2(t)e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2) &\leq (\alpha(\tau^*)^2(t) + \tau^*(t))e^{2\alpha t} \|\mathbf{u}_{ht}\|^2 \\ &+ \gamma(\tau^*)^2(t)e^{2\alpha t} \|\mathbf{u}_h\|_2 \|\mathbf{u}_{htt}\| + \delta(\tau^*)^2(t)e^{2\alpha t} \int_0^t \beta(t-s) \|\mathbf{u}_h(s)\|_2 \|\mathbf{u}_{htt}\| ds \\ &+ (\tau^*)^2(t)e^{2\alpha t} (|b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt})| + |b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt})| + \|f_t\| \|\mathbf{u}_{htt}\|) \end{aligned}$$

Use (3.8) to find that

$$|b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt})| + |b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt})| \leq \frac{1}{4} \|\mathbf{u}_{htt}\|^2 + K \|\mathbf{u}_{ht}\|_1^2 \|\mathbf{u}_h\|_2^2.$$

Now, using (3.9)-(3.10), we can easily deduce from (3.13) that

$$(3.14) \quad (\tau^*)^2 \|\mathbf{u}_{ht}\|_1^2 + \mu e^{-2\alpha t} \int_0^t (\tau^*)^2(s) e^{2\alpha s} \|\mathbf{u}_{hss}\|^2 ds \leq K.$$

We set $\phi_h = -\tau^*(t) e^{2\alpha t} \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}$ in (3.12). From (3.8), we see that

$$b(\mathbf{u}_{ht}, \mathbf{u}_h, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}) \leq K \|\mathbf{u}_{ht}\|^{1/2} \|\mathbf{u}_{ht}\|_1^{1/2} \|\mathbf{u}_h\|_1 \|\mathbf{u}_{htt}\|_{-1}$$

and therefore

$$\begin{aligned} & \mu \frac{d}{dt} (\tau^*(t) e^{2\alpha t} \|\mathbf{u}_{ht}\|^2) + \tau^*(t) e^{2\alpha t} \|\mathbf{u}_{htt}\|_{-1}^2 \leq (2\alpha \tau^*(t) + 1) e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2 \\ & + C(\mu, \gamma) \tau^*(t) e^{2\alpha t} \|\nabla \mathbf{u}_h\|^2 + 2 \|\mathbf{f}_t\|^2 + C(\mu, \delta) \left(\int_0^t \beta(t-s) e^{\alpha t} \|\tilde{\Delta}_h \mathbf{u}_h(s)\| ds \right)^2 \\ & + C(\mu) \tau^*(t) e^{2\alpha t} \left(\|\nabla \mathbf{u}_h\|^2 \|\mathbf{u}_{ht}\|^2 + \|\nabla \mathbf{u}_{ht}\|^2 (1 + \|\mathbf{u}_h\| \|\nabla \mathbf{u}_h\|) \right). \end{aligned}$$

Integrate with respect to time and multiply by $e^{-2\alpha t}$ to conclude

$$(3.15) \quad \tau^*(t) \|\mathbf{u}_{ht}\|^2 + \mu e^{-2\alpha t} \int_0^t \tau^*(s) e^{2\alpha s} \|\mathbf{u}_{hss}\|_{-1}^2 ds \leq K.$$

Finally, we set $\phi_h = -e^{2\alpha t} \tilde{\Delta}_h^2 \mathbf{u}_{htt}$ in (3.12) and proceed as above to arrive at

$$(3.16) \quad \|\mathbf{u}_{ht}\|_{-1}^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\mathbf{u}_{hss}\|_{-2}^2 ds \leq K.$$

This completes the rest of the proof. □

The following semi-discrete error estimates are proved in [8].

Theorem 3.1. *Let Ω be a convex polygon and let the conditions (A1)-(A2) and (B1)-(B2) be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h} \in \mathbf{J}_h$ with $\mathbf{u}_{0h} = P_h \mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbf{J}_1$. Then, there exists a positive constant C such that for $0 < T < \infty$ with $t \in (0, T]$*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h \|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\| \leq C e^{Ct} h^2 t^{-1/2}.$$

Moreover, under the assumption of the uniqueness condition, that is,

$$(3.17) \quad \frac{N}{\nu^2} \|\mathbf{f}\|_\infty < 1 \quad \text{and} \quad N = \sup_{\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{u}, \mathbf{v}, \mathbf{w})}{\|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|},$$

where $\nu = \mu + \frac{\gamma}{\delta}$ and $\|\mathbf{f}\|_\infty := \|\mathbf{f}\|_{L^\infty(0, \infty; \mathbf{L}^2(\Omega))}$ then we have the following uniform estimate:

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq C h^2 t^{-1/2}.$$

4 Backward Euler Method

For time discretization, we state below some notations. Let k , $0 < k < 1$, be the time step and let $t_n = nk$, $n \geq 0$. We define for a sequence $\{\phi^n\}_{n \geq 0} \subset \mathbf{J}_h$,

$$\partial_t \phi^n = \frac{1}{k} (\phi^n - \phi^{n-1}).$$

For continuous function $\mathbf{v}(t)$, we set $\mathbf{v}_n = \mathbf{v}(t_n)$. Since backward Euler method is of first order in time, we choose the right rectangle rule to approximate the integral term in (3.4) as:

$$(4.1) \quad q_r^n(\phi) = k \sum_{j=1}^n \beta_{n-j} \phi^j \approx \int_0^{t_n} \beta(t_n - s) \phi(s) ds$$

where $\beta_{n-j} = \beta(t_n - t_j)$. With $w_{nj} = k\beta(t_n - t_j)$, it is observed that the the right rectangle rule is positive in the sense that

$$(4.2) \quad k \sum_{i=1}^n q_r^i(\phi) \phi^i = k \sum_{i=1}^n k \sum_{j=0}^i \omega_{ij} \phi^j \phi^i \geq 0, \quad \phi = (\phi^0, \dots, \phi^N)^T.$$

For positivity of the rectangle rule with $\omega_{n0} = 0$, we refer to McLean and Thomée [14]. Note that the error incurred due to right rectangle rule in approximating the integral term is

$$(4.3) \quad \begin{aligned} \varepsilon_r^n(\phi) &:= \int_0^{t_n} \beta(t_n - s) \phi(s) ds - k \sum_{j=1}^n \beta_{n-j} \phi^j \\ &\leq Kk \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left| \frac{\partial}{\partial s} (\beta(t_n - s) \phi(s)) \right| ds. \end{aligned}$$

We present here a discrete version of integration by parts. For sequences $\{a_i\}$ and $\{b_i\}$ of real numbers, the following summation by parts holds

$$(4.4) \quad k \sum_{j=1}^i a_j b_j = a_i \hat{b}_i - k \sum_{j=1}^{i-1} (\partial_t a_{j+1}) \hat{b}_j,$$

where $\hat{b}_i := k \sum_{j=1}^i b_j$.

We describe below the backward Euler scheme for the semidiscrete Oldroyd problem (3.2): Find $\{\mathbf{U}^n\}_{n \geq 0} \in \mathbf{H}_h$ and $\{P^n\}_{n \geq 1} \in L_h$ as solutions of the recursive nonlinear algebraic equations ($n \geq 1$) :

$$(4.5) \quad \left. \begin{aligned} (\partial_t \mathbf{U}^n, \phi_h) + \mu a(\mathbf{U}^n, \phi_h) &+ a(q_r^n(\mathbf{U}), \phi_h) = (P^n, \nabla \cdot \phi_h) \\ &+ (\mathbf{f}^n, \phi_h) - b(\mathbf{U}^n, \mathbf{U}^n, \phi_h) \quad \forall \phi_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{U}^n, \chi_h) &= 0 \quad \forall \chi_h \in L_h, \quad n \geq 0. \end{aligned} \right\}$$

We choose $\mathbf{U}^0 = \mathbf{u}_{0h} = P_h \mathbf{u}_0$. Now, for $\phi_h \in \mathbf{J}_h$, we seek $\{\mathbf{U}^n\}_{n \geq 0} \in \mathbf{J}_h$ such that

$$(4.6) \quad (\partial_t \mathbf{U}^n, \phi_h) + \mu a(\mathbf{U}^n, \phi_h) + a(q_r^n(\mathbf{U}), \phi_h) = (\mathbf{f}^n, \phi_h) - b(\mathbf{U}^n, \mathbf{U}^n, \phi_h) \quad \forall \phi_h \in \mathbf{J}_h.$$

Using variant of Brouwer fixed point theorem and standard uniqueness arguments, it is easy to show that the discrete problem (4.6) is well-posed. For a proof, we refer to [7]. Below we prove *a priori* bounds for the discrete solutions $\{\mathbf{U}^n\}_{n \geq 0}$.

Lemma 4.1. *Let $0 < \alpha < \min\{\delta, \frac{\mu\lambda_1}{2}\}$ and $k_0 > 0$ be such that for $0 < k < k_0$*

$$1 + \left(\frac{\mu\lambda_1}{2}\right)k \geq e^{\alpha k}.$$

Further, let $\mathbf{U}^0 = \mathbf{u}_{0h} = P_h \mathbf{u}_0$ with $\mathbf{u}_0 \in \mathbf{J}_1$. Then, the discrete solution \mathbf{U}^N , $N \geq 1$ of (4.6) satisfies the following estimates:

$$(4.7) \quad \|\mathbf{U}^N\|^2 + \Gamma_1 e^{-\alpha t_N} k \sum_{n=1}^N e^{\alpha t_n} \|\nabla \mathbf{U}^n\|^2 \leq C \left(e^{-\alpha t_N} \|\mathbf{U}^0\|^2 + \|\mathbf{f}\|_\infty^2 \right),$$

where $\|\mathbf{f}\|_\infty = \|\mathbf{f}\|_{L^\infty(\mathbf{L}^2)}$, and

$$\Gamma_1 = \left(e^{-\alpha k} \mu - 2 \left(\frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1} \right).$$

Proof. Setting $\tilde{\mathbf{U}}^n = e^{\alpha t_n} \mathbf{U}^n$, we rewrite (4.6), for $\phi_h \in \mathbf{J}_h$, as

$$(4.8) \quad e^{\alpha t_n} (\partial_t \mathbf{U}^n, \phi_h) + \mu a(\tilde{\mathbf{U}}^n, \phi_h) + e^{-\alpha t_n} b_h(\tilde{\mathbf{U}}^n, \tilde{\mathbf{U}}^n, \phi_h) + e^{\alpha t_n} a(q_r^n(\mathbf{U}), \phi_h) = (\tilde{\mathbf{f}}^n, \phi_h).$$

Note that

$$e^{\alpha t_n} \partial_t \mathbf{U}^n = e^{\alpha k} \partial_t \tilde{\mathbf{U}}^n - \left(\frac{e^{\alpha k} - 1}{k} \right) \tilde{\mathbf{U}}^n.$$

On substituting this in (4.8) and then multiplying the resulting equation by $e^{-\alpha k}$, we obtain

$$(4.9) \quad (\partial_t \tilde{\mathbf{U}}^n, \phi_h) - \left(\frac{1 - e^{-\alpha k}}{k} \right) (\tilde{\mathbf{U}}^n, \phi_h) + e^{-\alpha k} \mu a(\tilde{\mathbf{U}}^n, \phi_h) + e^{-\alpha t_{n+1}} b(\tilde{\mathbf{U}}^n, \tilde{\mathbf{U}}^n, \phi_h) \\ + \gamma e^{-\alpha k} \sum_{i=1}^n e^{-(\delta - \alpha)(t_n - t_i)} a(\tilde{\mathbf{U}}^i, \phi_h) = e^{-\alpha k} (\tilde{\mathbf{f}}^n, \phi_h).$$

Put $\phi_h = \tilde{\mathbf{U}}^n$ in (4.9) and observe that

$$(\partial_t \phi^n, \phi^n) = \frac{1}{k} (\phi^n - \phi^{n-1}, \phi^n) \geq \frac{1}{2k} (\|\phi^n\|^2 - \|\phi^{n-1}\|^2) = \frac{1}{2} \partial_t \|\phi^n\|^2,$$

and that the nonlinear term vanishes. Also use $\|\tilde{\mathbf{U}}^n\|^2 \leq \frac{1}{\lambda_1} \|\nabla \tilde{\mathbf{U}}^n\|^2$ to obtain

$$(4.10) \quad \frac{1}{2} \partial_t \|\tilde{\mathbf{U}}^n\|^2 + \left(e^{-\alpha k} \mu - \left(\frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1} \right) \|\nabla \tilde{\mathbf{U}}^n\|^2 \\ + \gamma e^{-\alpha k} k \sum_{i=1}^n e^{-(\delta - \alpha)(t_n - t_i)} a(\tilde{\mathbf{U}}^i, \tilde{\mathbf{U}}^n) \leq e^{-\alpha k} \|\tilde{\mathbf{f}}^n\| \|\tilde{\mathbf{U}}^n\|.$$

The right-hand side of (4.10) can be estimated as

$$\frac{1}{2} e^{-\alpha k} \mu \|\nabla \tilde{\mathbf{U}}^n\|^2 + \frac{1}{2\mu\lambda_1} e^{-\alpha k} \|\tilde{\mathbf{f}}^n\|^2,$$

so as to obtain from (4.10)

$$(4.11) \quad \partial_t \|\tilde{\mathbf{U}}^n\|^2 + \left(e^{-\alpha k} \mu - 2 \left(\frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1} \right) \|\nabla \tilde{\mathbf{U}}^n\|^2 \\ + 2\gamma e^{-\alpha k} k \sum_{i=1}^n e^{-(\delta - \alpha)(t_n - t_i)} a(\tilde{\mathbf{U}}^i, \tilde{\mathbf{U}}^n) \leq \frac{1}{\mu\lambda_1} e^{-\alpha k} \|\tilde{\mathbf{f}}^n\|^2.$$

With $0 < \alpha < \min\{\delta, \frac{\mu\lambda_1}{2}\}$, we choose $k_0 > 0$ such that for $0 < k < k_0$

$$1 + \left(\frac{\mu\lambda_1}{2} \right) k \geq e^{\alpha k}.$$

This guarantees that $e^{-\alpha k} \mu - 2 \left(\frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1} \geq 0$. Multiply (4.11) by k and then sum over $n = 1$ to N . The resulting double sum is non-negative and hence, we obtain

$$(4.12) \quad \|\tilde{\mathbf{U}}^N\|^2 + \Gamma_1 k \sum_{n=1}^N \|\nabla \tilde{\mathbf{U}}^n\|^2 \leq \|\mathbf{U}^0\|^2 + \frac{\|\mathbf{f}\|_\infty^2}{\mu\lambda_1} e^{-\alpha k} k \sum_{n=1}^N e^{2\alpha t_n}.$$

Note that using geometric series, we find that

$$(4.13) \quad k \sum_{n=1}^N e^{2\alpha t_n} = e^{2\alpha k} \frac{k}{e^{2\alpha k} - 1} e^{2\alpha t_N} = e^{2\alpha(k - k^*)} e^{2\alpha t_N},$$

for some k^* in $(0, k)$. On substituting (4.13) in (4.12), multiply through out by $e^{-\alpha t_N}$ to complete the rest of the proof. \square

In order to obtain uniform (in time) estimate for the discrete solution \mathbf{U}^n in Dirichlet norm, we introduce the following notation:

$$(4.14) \quad \mathbf{U}_\beta^n = k \sum_{j=1}^n \beta_{nj} \mathbf{U}^j, n > 0; \quad \mathbf{U}_\beta^0 = 0,$$

and rewrite (4.6), for $\phi_h \in \mathbf{J}_h$, as

$$(4.15) \quad (\partial_t \mathbf{U}^n, \phi_h) + \mu a(\mathbf{U}^n, \phi_h) + b_h(\mathbf{U}^n, \mathbf{U}^n, \phi_h) + a(\mathbf{U}_\beta^n, \phi_h) = (\mathbf{f}^n, \phi_h).$$

Note that

$$(4.16) \quad \mathbf{U}_\beta^n = k\gamma \mathbf{U}^n + e^{-\delta k} \mathbf{U}_\beta^{n-1},$$

and therefore

$$(4.17) \quad \begin{aligned} \partial_t \mathbf{U}_\beta^n &= \frac{1}{k} (\mathbf{U}_\beta^n - \mathbf{U}_\beta^{n-1}) = \frac{1}{k} (k\gamma \mathbf{U}^n + e^{-\delta k} \mathbf{U}_\beta^{n-1} - \mathbf{U}_\beta^{n-1}) \\ &= \gamma \mathbf{U}^n - \frac{(1 - e^{-\delta k})}{k} \mathbf{U}_\beta^{n-1}. \end{aligned}$$

Lemma 4.2. *Let $0 < \alpha < \min(\delta, \mu\lambda_1/2)$, $\mathbf{U}^0 = P_h \mathbf{u}_0$ and $k_0 > 0$ be such that for $0 < k < k_0$*

$$1 + \left(\frac{\mu\lambda_1}{2}\right)k \geq e^{\alpha k}.$$

Then, the discrete solution \mathbf{U}^n , $n \geq 1$ of (4.6) satisfies the following uniform estimates:

$$(4.18) \quad \|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2 \leq e^{-2\alpha n} \|\mathbf{U}^0\|^2 + \left(\frac{1 - e^{-2\alpha n}}{\alpha\mu\lambda_1}\right) \|\mathbf{f}\|_\infty^2 = M_{11}^2,$$

and

$$(4.19) \quad k \sum_{n=m}^{m+l} (\mu \|\nabla \mathbf{U}^n\|^2 + \frac{\delta}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2) \leq M_{11}^2 + \frac{l}{\mu\lambda_1} \|\mathbf{f}\|_\infty^2 = M_{12}^2(l),$$

where \mathbf{U}_β^n is given by (4.14).

Proof. Take $\phi_h = \mathbf{U}^n$ in (4.15) and from (4.17), we find that

$$a(\mathbf{U}_\beta^n, \mathbf{U}^n) = \frac{e^{-\delta k}}{\gamma} a(\mathbf{U}_\beta^n, \partial_t \mathbf{U}_\beta^n) + \frac{(1 - e^{-\delta k})}{k\gamma} \|\nabla \mathbf{U}_\beta^n\|^2.$$

Using mean value theorem, we observe that

$$\frac{(1 - e^{-\delta k})}{k} = \delta e^{-\delta k^*} \geq \delta e^{-\delta k}, \quad k^* \in (0, k).$$

Therefore, we obtain from (4.15)

$$(4.20) \quad \partial_t (\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2) + \mu \|\nabla \mathbf{U}^n\|^2 + \frac{2\delta e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2 \leq \frac{1}{\mu\lambda_1} \|\mathbf{f}^n\|^2.$$

As $0 < \alpha < \min\{\delta, \mu\lambda_1/2\}$, we now find that

$$(4.21) \quad \partial_t (\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2) + 2\alpha (\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2) \leq \frac{1}{\mu\lambda_1} \|\mathbf{f}^n\|^2.$$

Multiply the inequality (4.21) by $e^{\alpha_0 t_{n-1}}$ for some $\alpha_0 > 0$ and note that

$$(4.22) \quad \begin{aligned} \partial_t (e^{\alpha_0 t_n} \phi^n) &= e^{\alpha_0 t_{n-1}} \left\{ \partial_t \phi^n + \frac{e^{\alpha_0 k} - 1}{k} \phi^n \right\} \\ &\leq e^{\alpha_0 t_{n-1}} \left\{ \partial_t \phi^n + 2\alpha \phi^n \right\}. \end{aligned}$$

With the assumption on the time step k , that is, $0 < k < k_0$, and for given α , we can always choose α_0 such that

$$(4.23) \quad 1 + 2\alpha k \geq e^{\alpha_0 k}.$$

Observe that $\alpha_0 < 2\alpha$. Therefore, we obtain from (4.21)

$$\partial_t \left(e^{\alpha_0 t_n} \left(\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2 \right) \right) \leq \frac{e^{\alpha_0 t_{n-1}}}{\mu \lambda_1} \|\mathbf{f}\|_\infty^2.$$

Multiply by k and sum over 1 to n and then multiply the resulting inequality by $e^{-\alpha_0 t_n}$. Observe that $\mathbf{U}_\beta^0 = 0$ by definition. This results in the first estimate (4.18). For the second estimate (4.19), we multiply (4.20) by k , sum over m to $m+l$ with $m, l \in \mathcal{N}$ and use (4.18) to complete the rest of the proof. \square

Lemma 4.3. *Under the assumptions of Lemma 4.2, the discrete solution \mathbf{U}^n , $n \geq 1$ of (4.6) satisfies the following uniform estimates:*

$$(4.24) \quad \|\nabla \mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\tilde{\Delta}_h \mathbf{U}_\beta^n\|^2 \leq K.$$

Proof. Set $\phi_h = -\tilde{\Delta}_h \mathbf{U}^n$ in (4.15) and as in the Lemma 4.2, we now obtain

$$(4.25) \quad \begin{aligned} \partial_t \left(\|\nabla \mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\tilde{\Delta}_h \mathbf{U}_\beta^n\|^2 \right) + \mu \|\tilde{\Delta}_h \mathbf{U}^n\|^2 + \frac{2\delta}{\gamma} \|\nabla \mathbf{U}_\beta^n\|^2 &\leq \|\mathbf{f}^n\| \|\tilde{\Delta}_h \mathbf{U}^n\| \\ &+ |b_h(\mathbf{U}^n, \mathbf{U}^n, -\tilde{\Delta}_h \mathbf{U}^n)|. \end{aligned}$$

Use Lemma 3.1 to arrive at

$$(4.26) \quad \begin{aligned} \partial_t \left(\|\nabla \mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\tilde{\Delta}_h \mathbf{U}_\beta^n\|^2 \right) + \frac{4\mu}{3} \|\tilde{\Delta}_h \mathbf{U}^n\|^2 + \frac{2\delta}{\gamma} \|\tilde{\Delta}_h \mathbf{U}_\beta^n\|^2 \\ \leq \frac{3}{\mu} \|\mathbf{f}\|_\infty^2 + \left(\frac{9}{2} \right)^3 M_{11}^2 \|\nabla \mathbf{U}^n\|^4. \end{aligned}$$

For some $\alpha_0 > 0$, we find that

$$(4.27) \quad \alpha_0 \|\nabla \mathbf{U}^n\|^2 \leq \frac{\mu}{3} \|\tilde{\Delta}_h \mathbf{U}^n\|^2 + \frac{3}{4\mu} \alpha_0^2 \|\mathbf{U}^n\|^2.$$

Define

$$(4.28) \quad g^n = \min \left\{ \alpha_0 + \mu \lambda_1 - \left(\frac{9}{2\mu} \right)^3 M_{11}^2 \|\nabla \mathbf{U}^n\|^2, 2\delta \right\}.$$

With $E^n := \|\nabla \mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma} \|\tilde{\Delta}_h \mathbf{U}_\beta^n\|^2$, we rewrite (4.26) as

$$(4.29) \quad \partial_t E^n + g^n E^n \leq \frac{3}{\mu} \|\mathbf{f}\|_\infty^2 + \frac{3}{4\mu} \alpha_0^2 \|\mathbf{U}^n\|^2 = K_{11}.$$

Let $\{n_i\}_{i \in \mathbb{N}}$ and $\{\bar{n}_i\}_{i \in \mathbb{N}}$ be two subsequences of natural numbers such that

$$g^{n_i} = \alpha_0 + \mu \lambda_1 - \left(\frac{9}{2\mu} \right)^3 M_{11}^2 \|\nabla \mathbf{U}^{n_i}\|^2, \quad g^{\bar{n}_i} = 2\delta, \quad \forall i.$$

If for some n ,

$$g^n = \alpha_0 + \mu \lambda_1 - \left(\frac{9}{2\mu} \right)^3 M_{11}^2 \|\nabla \mathbf{U}^n\|^2 = 2\delta$$

then without loss of generality, we assume that $n \in \{\bar{n}_i\}$ so as to make the two subsequence $\{n_i\}$ and $\{\bar{n}_i\}$ disjoint. Now for $m, l \in \mathbb{N}$, we write

$$(4.30) \quad \begin{aligned} k \sum_{n=m}^{m+l} g^n &= k \sum_{n=m_1}^{m_{l_1}} g^n + k \sum_{n=\bar{m}_1}^{\bar{m}_{l_2}} g^n \\ &= k \sum_{n=m_1}^{m_{l_1}} \left(\alpha_0 + \mu \lambda_1 - \left(\frac{9}{2\mu} \right)^3 M_{11}^2 \|\nabla \mathbf{U}^n\|^2 \right) + k \sum_{n=\bar{m}_1}^{\bar{m}_{l_2}} 2\delta. \end{aligned}$$

Here, $m_1, m_2, \dots, m_{l_1} \in \{n_i\} \cap \{m, m+1, \dots, m+l\}$ and $\bar{m}_1, \bar{m}_2, \dots, \bar{m}_{l_2} \in \{\bar{n}_i\} \cap \{m, m+1, \dots, m+l\}$ such that $l_1 + l_2 = l+1$. Note that l_1 or l_2 could be 0. Using Lemma 4.2, we observe that

$$\left(\frac{9}{2\mu}\right)^3 k \sum_{n=m}^{m+l} M_{11}^2 \|\nabla \mathbf{U}^n\|^2 \leq \frac{9^3 M_{11}^2}{2^3 \mu^3} k \sum_{n=m}^{m+l} \|\nabla \mathbf{U}^n\|^2 \leq \frac{9^3 M_{11}^2}{2^3 \mu^4} M_{12}^2(l) = K_{12}(l).$$

Therefore, from (4.30), we find that

$$k \sum_{n=m}^{m+l} g^n \geq (kl_1)(\alpha_0 + \mu\lambda_1) - K_{12}(l_1) + 2\delta(kl_2).$$

We choose α_0 such that $(kl_1)(\alpha_0 + \mu\lambda_1) - K_{12}(l_1) = 2\delta(kl_1)$ to arrive at

$$(4.31) \quad k \sum_{n=m}^{m+l} g^n \geq 2\delta t_{l+1}.$$

By definition of g^n , we have equality in (4.31) and in fact, $g^n = 2\delta$. Now from (4.29), we obtain

$$\partial_t E^n + 2\delta E^n \leq K_{11}.$$

As in (4.22), we can choose $0 < \alpha_{01} < \alpha \leq \delta$ such that

$$\partial_t (e^{\alpha_{01} t_n} E^n) \leq e^{\alpha_{01} t_{n-1}} (\partial_t E^n + 2\delta E^n) \leq K_{11} e^{\alpha_{01} t_{n-1}}.$$

Multiply by k and sum over 1 to n . Observe that $E^0 = \|\nabla \mathbf{U}^0\|^2$. Finally, multiply the resulting inequality by $e^{-\alpha_{01} t_n}$ to find that

$$E^n \leq e^{-\alpha_{01} t_n} \|\nabla \mathbf{U}^0\|^2 + K.$$

This completes the rest of the proof. \square

Remark 4.1. As a consequence of the Lemma 4.3, the following a priori bound is valid:

$$(4.32) \quad \tau * (t_n) \|\tilde{\Delta}_h \mathbf{U}^n\|^2 \leq K.$$

5 A Priori Error Estimate

In this section, we discuss error estimate of the backward Euler method for the Oldroyd model (1.1)-(1.3). For the error analysis, we set, for fixed $n \in \mathbb{N}$, $1 < n \leq N$, $\mathbf{e}_n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$. We now rewrite (3.4) at $t = t_n$ and subtract the resulting one from (4.6) to obtain

$$(5.1) \quad (\partial_t \mathbf{e}_n, \phi_h) + \mu a(\mathbf{e}_n, \phi_h) + a(q_r^n(\mathbf{e}), \phi_h) = E^n(\mathbf{u}_h)(\phi_h) + \varepsilon_a^n(\mathbf{u}_h)(\phi_h) + \Lambda_h^n(\phi_h),$$

where,

$$(5.2) \quad \begin{aligned} E^n(\mathbf{u}_h)(\phi_h) &= (\mathbf{u}_{ht}^n, \phi_h) - (\partial_t \mathbf{u}_h^n, \phi_h) = (\mathbf{u}_{ht}^n, \phi_h) - \frac{1}{k} \int_{t_{n-1}}^{t_n} (\mathbf{u}_{hs}, \phi_h) ds \\ &= \frac{1}{2k} \int_{t_{n-1}}^{t_n} (t - t_{n-1})(\mathbf{u}_{htt}, \phi_h) dt, \end{aligned}$$

$$(5.3) \quad \varepsilon_a^n(\mathbf{u}_h)(\phi_h) = a(\mathbf{u}_{h,\beta}(t_n), \phi_h) - a(q_r^n(\mathbf{u}_h), \phi_h) = a(\varepsilon_r^n(\mathbf{u}_h), \phi_h),$$

and

$$(5.4) \quad \begin{aligned} \Lambda_h^n(\phi_h) &= b(\mathbf{u}_h^n, \mathbf{u}_h^n, \phi_h) - b(\mathbf{U}^n, \mathbf{U}^n, \phi_h) \\ &= -b(\mathbf{u}_h^n, \mathbf{e}_n, \phi_h) - b(\mathbf{e}_n, \mathbf{u}_h^n, \phi_h) - b(\mathbf{e}_n, \mathbf{e}_n, \phi_h). \end{aligned}$$

In order to dissociate the effect of nonlinearity, we first linearized the discrete problem (4.6), and introduce $\{\mathbf{V}^n\}_{n \geq 1} \in \mathbf{J}_h$ as solutions of the following linearized problem:

$$(5.5) \quad (\partial_t \mathbf{V}^n, \phi_h) + \mu a(\mathbf{V}^n, \phi_h) + a(q_r^n(\mathbf{V}), \phi_h) = (\mathbf{f}^n, \phi_h) - b(\mathbf{u}_h^n, \mathbf{u}_h^n, \phi_h) \quad \forall \phi_h \in \mathbf{J}_h,$$

given $\{\mathbf{U}^n\}_{n \geq 1} \in \mathbf{J}_h$ as solution of (4.6). It is easy to check the existence and uniqueness of $\{\mathbf{V}^n\}_{n \geq 1} \in \mathbf{J}_h$.

We now split the error as:

$$(5.6) \quad \mathbf{e}_n := \mathbf{U}^n - \mathbf{u}_h^n = (\mathbf{U}^n - \mathbf{V}^n) - (\mathbf{u}_h^n - \mathbf{V}^n) =: \boldsymbol{\eta}_n - \boldsymbol{\xi}_n.$$

The following equations are satisfied by $\boldsymbol{\xi}_n$ and $\boldsymbol{\eta}_n$, respectively:

$$(5.7) \quad (\partial_t \boldsymbol{\xi}_n, \phi_h) + \mu a(\boldsymbol{\xi}_n, \phi_h) + a(q_r^n(\boldsymbol{\xi}), \phi_h) = -E^n(\mathbf{u}_h)(\phi_h) - \varepsilon_a^n(\mathbf{u}_h)(\phi_h)$$

and

$$(5.8) \quad (\partial_t \boldsymbol{\eta}_n, \phi_h) + \mu a(\boldsymbol{\eta}_n, \phi_h) + a(q_r^n(\boldsymbol{\eta}), \phi_h) = \Lambda_h^n(\phi_h).$$

Below, we prove the following Lemma for our subsequent use.

Lemma 5.1. *Let $r, s \in \{0, 1\}$, $\tau_i = \min\{1, t_i\}$ and α as defined in Lemma 4.1. Then, with E^n and ε_a^n defined, respectively, as (5.2) and (5.3), the following estimate holds for $n = 1, \dots, N$ and for $\{\phi_h^i\}_i$ in \mathbf{J}_h :*

$$(5.9) \quad \begin{aligned} & 2k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} \left(E^i(\mathbf{u}_h)(\phi_h^i) + \varepsilon_a^i(\mathbf{u}_h)(\phi_h^i) \right) \\ & \leq K k^{(1+s-r)/2} (1 + \log \frac{1}{k})^{(1-r)/2} \left(k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} \|\phi_h^i\|_{1-r}^2 \right)^{1/2}. \end{aligned}$$

Proof. From (5.2), we observe that

$$\begin{aligned} & 2k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} E^i(\mathbf{u}_h)(\phi_h^i) \\ & \leq \left[k^{-1} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \tau_i^{s/2} e^{\alpha(t_i - t_n)} (t - t_{i-1}) \|\mathbf{u}_{htt}\|_{r-1} dt \right)^2 \right]^{1/2} \left[k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} \|\phi_h^i\|_{1-r}^2 \right]^{1/2}. \end{aligned}$$

Using (3.11), we find

$$(5.10) \quad \begin{aligned} & \left[k^{-1} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \tau_i^{s/2} e^{\alpha(t_i - t_n)} (t - t_{i-1}) \|\mathbf{u}_{htt}\|_{r-1} dt \right)^2 \right]^{1/2} \\ & \leq \left[k^{-1} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \tau_i^s \tau^{-(r+1)} (t - t_{i-1})^2 e^{2\alpha(t_i - t)} dt \right]^{1/2} \left[e^{-2\alpha t_n} \int_0^{t_n} \tau^{(r+1)} e^{2\alpha t} \|\mathbf{u}_{htt}\|_{r-1}^2 dt \right]^{1/2} \\ & \leq K e^{\alpha k} \left[k^{-1} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \tau_i^s \tau^{-(r+1)} (t - t_{i-1})^2 dt \right]^{1/2}. \end{aligned}$$

It is now easy to calculate the remaining part for various values of r, s . For the sake of completeness, we present below the case when $r = s = 0$.

$$\begin{aligned} & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} t^{-1} (t - t_{i-1})^2 dt \leq \int_0^k t dt + k^2 \sum_{i=2}^n \int_{t_{i-1}}^{t_i} t^{-1} dt \\ & \leq K k^2 (1 + \log \frac{1}{k}). \end{aligned}$$

This completes the proof of the first half. For the remaining part, we observe from (5.3) and (4.3) that

$$(5.11) \quad \begin{aligned} & 2k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} \varepsilon_a^i(\mathbf{u}_h)(\phi_h^i) \leq \left[k \sum_{i=1}^n \tau_i^s e^{2\alpha(t_i - t_n)} \|\phi_h^i\|_{1-r}^2 \right]^{1/2} \times \\ & \left[4k \sum_{i=1}^n \left(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} \tau_i^{s/2} e^{\alpha(t_i - t_n)} (t - t_{j-1}) \beta(t_i - t) \{ \delta \|\mathbf{u}_h\|_{r+1} + \|\mathbf{u}_{ht}\|_{r+1} \} dt \right)^2 \right]^{1/2}. \end{aligned}$$

In Lemma 3.2, we find that the estimates of $\|\mathbf{u}_{htt}\|_{r-1}$ and $\|\mathbf{u}_{ht}\|_{r+1}$ are similar, in fact, the powers of t_i are same. Therefore, the right-hand side of (5.11) involving $\|\mathbf{u}_{ht}\|_{r+1}$ can be estimated similarly as in (5.10). The terms involving $\|\mathbf{u}_h\|_{r+1}$ are clearly easy to estimate. But for the sake of completeness, we provide the case, when $r = s = 0$.

$$\begin{aligned}
& 4\delta^2 k \sum_{i=1}^n \left(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} e^{\alpha(t_i-t_n)} (t-t_{j-1}) \beta(t_i-t) \|\nabla \mathbf{u}_h\| dt \right)^2 \\
& \leq 4\gamma^2 \delta^2 e^{-2\alpha t_n} k^3 \sum_{i=1}^n e^{-2(\delta-\alpha)t_i} \left(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} e^{(\delta-\alpha)t} \|\nabla \hat{\mathbf{u}}_h\| dt \right)^2 \\
& \leq 4\gamma^2 \delta^2 e^{-2\alpha t_n} k^3 \sum_{i=1}^n e^{-2(\delta-\alpha)t_i} \left(\int_0^{t_i} e^{2(\delta-\alpha)s} ds \right) \left(\int_0^{t_i} \|\nabla \hat{\mathbf{u}}_h(s)\|^2 ds \right) \\
& \leq \frac{2\gamma^2 \delta^2}{2(\delta-\alpha)} e^{-2\alpha t_n} k^3 \sum_{i=1}^n e^{2(\delta-\alpha)t_i} (K e^{2\alpha t_i}) \leq K k^3 e^{2\delta k}.
\end{aligned}$$

This completes the rest of the proof. \square

Lemma 5.2. Assume (A1)-(A2) and a spatial discretization scheme that satisfies conditions (B1)-(B2) and (B4). Let $0 < \alpha < \min\{\delta, \mu\lambda_1\}$, and

$$1 + (\mu\lambda_1)k > e^{2\alpha k}$$

which holds for $0 < k < k_0$, $k_0 > 0$. Further, assume that $\mathbf{u}_h(t)$ and \mathbf{V}^n satisfy (3.4) and (5.5), respectively. Then, there is a positive constant K such that

$$(5.12) \quad \|\xi_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\xi_i\|_1^2 \leq Kk(1 + \log \frac{1}{k}),$$

$$(5.13) \quad \|\xi_n\|_1^2 + k \sum_{i=1}^n \{\|\xi_i\|_2^2 + \|\partial_t \xi_i\|^2\} \leq K.$$

Proof. For $n = i$, we put $\phi_h = \xi_i$ in (5.7) and with the observation

$$(\partial_t \xi_i, \xi_i) = \frac{1}{2k} (\xi_i - \xi_{i-1}, \xi_i) \geq \frac{1}{2k} (\|\xi_i\|^2 - \|\xi_{i-1}\|^2) = \frac{1}{2} \partial_t \|\xi_i\|^2,$$

we find that

$$(5.14) \quad \partial_t \|\xi_i\|^2 + 2\mu \|\nabla \xi_i\|^2 + a(q_r^i(\xi), \xi_i) \leq -2E^i(\mathbf{u}_h)(\xi_i) - 2\varepsilon_a^i(\mathbf{u}_h)(\xi_i).$$

Multiply (5.14) by $ke^{2\alpha t_i}$ and sum over $1 \leq i \leq n \leq N$ to obtain

$$\begin{aligned}
(5.15) \quad \|\tilde{\xi}_n\|^2 - \sum_{i=1}^{n-1} (e^{2\alpha k} - 1) \|\tilde{\xi}_i\|^2 + 2\mu k \sum_{i=1}^n \|\nabla \tilde{\xi}_i\|^2 & \leq -2k \sum_{i=1}^n e^{2\alpha t_i} \left\{ E^i(\mathbf{u}_h)(\xi_i) + \varepsilon_a^i(\mathbf{u}_h)(\xi_i) \right\} \\
& \leq \mu k \sum_{i=1}^n \|\nabla \tilde{\xi}_i\|^2 + Kk(1 + \log \frac{1}{k}) e^{2\alpha t_{n+1}}.
\end{aligned}$$

Recall that $\tilde{v}(t) = e^{\alpha t} v(t)$. Note that we have dropped the quadrature term on the left hand-side of (5.14) after summation as it is non-negative. Finally, we have used Lemma 5.1 for $s = r = 0$. We note that for $0 < k < k_0$

$$\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1} > 0,$$

and hence,

$$(5.16) \quad \|\tilde{\xi}_n\|^2 + \left(\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1} \right) k \sum_{i=1}^n \|\nabla \tilde{\xi}_i\|^2 \leq Kk(1 + \log \frac{1}{k}) e^{2\alpha t_{n+1}}.$$

Multiply (5.16) by $e^{-2\alpha t_n}$ to establish (5.12). Next, for $n = i$, we put $\phi_h = -\tilde{\Delta}_h \xi_i$ in (5.7) and follow as above to obtain the first part of (5.13), that is,

$$\|\xi_n\|_1^2 + k \sum_{i=1}^n \|\xi_i\|_2^2 \leq K.$$

Here, we have used (5.9) for $s = 0, r = 1$ with $\alpha = 0$ replacing ϕ_h^i by $\tilde{\Delta}_h \xi_i$.

Finally, for $n = i$, we put $\phi_h = \partial_t \xi_i$ in (5.7) to find that

$$(5.17) \quad 2\|\partial_t \xi_i\|^2 + \mu \partial_t \|\xi_i\|_1^2 \leq -2a(q_r^i(\xi), \partial_t \xi_i) - 2E^i(\mathbf{u}_h)(\partial_t \xi_i) - 2\varepsilon_a^i(\mathbf{u}_h)(\partial_t \xi_i).$$

Multiply (5.17) by $ke^{2\alpha t_i}$ and sum over $1 \leq i \leq n \leq N$. As has been done earlier, we can estimate the last two resulting terms on the right-hand side of (5.17) using (5.9) for $r = s = 0$ as

$$\frac{k}{2} \sum_{i=1}^n e^{2\alpha t_i} \|\partial_t \xi_i\|^2 + K.$$

The only difference is that the resulting double sum (the term involving q_r^i) is no longer non-negative and hence, we need to estimate it. Note that

$$(5.18) \quad \begin{aligned} 2k \sum_{i=1}^n e^{2\alpha t_i} a(q_r^i(\xi), \partial_t \xi_i) &= 2\gamma k^2 \sum_{i=1}^n \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} a(\tilde{\xi}_j, e^{\alpha t_i} \partial_t \xi_i) \\ &\leq \frac{k}{2} \sum_{i=1}^n e^{2\alpha t_i} \|\partial_t \xi_i\|^2 + K(\gamma) k \sum_{i=1}^n \left(k \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} \|\tilde{\Delta}_h \tilde{\xi}_j\| \right)^2. \end{aligned}$$

Using change of variable and change of order of double sum, we obtain

$$\begin{aligned} I &:= K(\gamma) k \sum_{i=1}^n \left(k \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} \|\tilde{\Delta}_h \tilde{\xi}_j\| \right)^2 \\ &\leq K(\gamma) k \sum_{i=1}^n \left(k \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} \right) \left(k \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} \|\tilde{\Delta}_h \tilde{\xi}_j\|^2 \right) \\ &\leq K(\alpha, \gamma) e^{(\delta-\alpha)k} k^2 \sum_{i=1}^n k \sum_{j=1}^i e^{-(\delta-\alpha)(t_i-t_j)} \|\tilde{\Delta}_h \tilde{\xi}_j\|^2. \end{aligned}$$

Introduce $l = i - j$ to find that

$$\begin{aligned} I &\leq K(\alpha, \gamma) e^{(\delta-\alpha)k} k^2 \sum_{i=1}^n k \sum_{l=i-1}^0 e^{-(\delta-\alpha)t_l} \|\tilde{\Delta}_h \tilde{\xi}_{i-l}\|^2 \quad \text{for } l = i - j \\ &= K(\alpha, \gamma) e^{(\delta-\alpha)k} k^2 \sum_{i=1}^n k \sum_{l=1}^i e^{-(\delta-\alpha)t_{l-1}} \|\tilde{\Delta}_h \tilde{\xi}_{i-l+1}\|^2. \end{aligned}$$

With change of summation, we now arrive at

$$\begin{aligned} I &\leq K(\alpha, \gamma) e^{(\delta-\alpha)k} k^2 \sum_{l=1}^n k \sum_{i=l}^n e^{-(\delta-\alpha)t_{l-1}} \|\tilde{\Delta}_h \tilde{\xi}_{i-l+1}\|^2 \\ &= K(\alpha, \gamma) e^{(\delta-\alpha)k} k^2 \sum_{l=1}^n k \sum_{j=1}^{n-l+1} e^{-(\delta-\alpha)t_{l-1}} \|\tilde{\Delta}_h \tilde{\xi}_j\|^2 \quad \text{for } j = i - l + 1 \\ (5.19) \quad &\leq K(\alpha, \gamma) e^{(\delta-\alpha)k} k \left(k \sum_{l=1}^{n-1} e^{-(\delta-\alpha)t_l} \right) \left(k \sum_{j=1}^n \|\tilde{\Delta}_h \tilde{\xi}_j\|^2 \right) \leq K. \end{aligned}$$

Combining (5.18)-(5.19), we find that

$$2k \sum_{i=1}^n e^{2\alpha t_i} a(q_r^i(\xi), \partial_t \xi_i) \leq \frac{k}{2} \sum_{i=1}^n e^{2\alpha t_i} \|\partial_t \xi_i\|^2 + K.$$

Therefore, we obtain

$$(5.20) \quad k \sum_{i=1}^n e^{2\alpha t_i} \|\partial_t \xi_i\|^2 + \mu \|\tilde{\xi}_n\|_1^2 \leq K + \mu k \sum_{i=1}^{n-1} \frac{(e^{2\alpha k} - 1)}{k} \|\tilde{\xi}_i\|_1^2.$$

Use (5.12) and the fact that $(e^{2\alpha k} - 1)/k \leq K(\alpha)$ to complete the rest of the proof. \square

Remark 5.1. We note that the restriction on k , that is $0 < k < k_0$ is not same in the Lemmas 4.1, and 5.2. Therefore, we take minimum of the k_0 's from Lemmas 4.1 and 5.2 and denote it as k_{00} , then for all k satisfying $0 < k < k_{00}$, all the result should hold.

Analogous to the semi-discrete case, we resort to duality argument to obtain optimal $L^2(\mathbf{L}^2)$ estimate. Consider the following backward problem: For a given \mathbf{W}_n and \mathbf{g}_i , let \mathbf{W}_i , $n \geq i \geq 1$ satisfy

$$(5.21) \quad (\phi_h, \partial_t \mathbf{W}_i) - \mu a(\phi_h, \mathbf{W}_i) - k \sum_{j=i}^n \beta(t_j - t_i) a(\phi_h, \mathbf{W}_j) = (\phi_h, e^{2\alpha t_i} \mathbf{g}_i), \phi_h \in \mathbf{J}_h.$$

The following *a priori* estimates are easy to derive.

Lemma 5.3. Let the assumptions (A2), (B1), (B2) and (B4) hold. Then, for $0 < k < k_0$, the following estimates hold under appropriate assumptions on \mathbf{W}_n and \mathbf{g} :

$$\|\mathbf{W}_0\|_r^2 + k \sum_{i=1}^n e^{-2\alpha t_i} \{\|\mathbf{W}_i\|_{r+1} + \|\partial_t \mathbf{W}_i\|_{r-1}\} \leq K \{\|\mathbf{W}_n\|_r^2 + k \sum_{i=1}^n e^{2\alpha t_i} \|\mathbf{g}_i\|_{r-1}^2\},$$

where $r \in \{0, 1\}$.

Lemma 5.4. Under the assumptions of Lemma 5.3, the following estimate holds:

$$(5.22) \quad e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\xi_i\|^2 \leq K k^2.$$

Proof. With

$$\mathbf{W}_n = (-\tilde{\Delta}_h)^{-1} \xi_n, \quad \mathbf{g}_i = \xi_i \quad \forall i$$

we choose $\phi_h = \xi_i$ in (5.21) and use (5.7) to obtain

$$(5.23) \quad \begin{aligned} e^{2\alpha t_i} \|\xi_i\|^2 &= (\xi_i, \partial_t \mathbf{W}_i) - \mu a(\xi_i, \mathbf{W}_i) - k \sum_{j=i}^n \beta(t_j - t_i) a(\xi_i, \mathbf{W}_j) \\ &= \partial_t (\xi_i, \mathbf{W}_i) - (\partial_t \xi_i, \mathbf{W}_{i-1}) - \mu a(\xi_i, \mathbf{W}_i) - k \sum_{j=i}^n \beta(t_j - t_i) a(\xi_i, \mathbf{W}_j) \\ &= \partial_t (\xi_i, \mathbf{W}_i) + k (\partial_t \xi_i, \partial_t \mathbf{W}_i) + k \sum_{j=1}^i \beta(t_i - t_j) a(\xi_j, \mathbf{W}_i) + E^i(\mathbf{u}_h)(\mathbf{W}_i) \\ &\quad + \varepsilon_a^i(\mathbf{u}_h)(\mathbf{W}_i) - k \sum_{j=i}^n \beta(t_j - t_i) a(\xi_i, \mathbf{W}_j). \end{aligned}$$

Multiply (5.23) by k and sum over $1 \leq i \leq n$. Observe that the resulting two double sums cancel out (change of order of double sum). Therefore, we find that

$$(5.24) \quad k \sum_{i=1}^n e^{2\alpha t_i} \|\xi_i\|^2 + \|\xi_n\|_{-1}^2 = k \sum_{i=1}^n [k (\partial_t \xi_i, \partial_t \mathbf{W}_i) + E^i(\mathbf{u}_h)(\mathbf{W}_i) + \varepsilon_a^i(\mathbf{u}_h)(\mathbf{W}_i)].$$

From (5.2), we observe that

$$(5.25) \quad \begin{aligned} k \sum_{i=1}^n E^i(\mathbf{u}_h)(\mathbf{W}_i) &\leq k \sum_{i=1}^n \frac{1}{2k} \int_{t_{i-1}}^{t_i} (s - t_{i-1}) \|\mathbf{u}_{hss}\|_{-2} \|\mathbf{W}_i\|_2 \\ &\leq \frac{k}{4} e^{\alpha k} \left(\int_0^{t_n} e^{2\alpha s} \|\mathbf{u}_{hss}\|_{-2}^2 ds \right)^{1/2} \left(k \sum_{i=1}^n e^{-2\alpha t_i} \|\mathbf{W}_i\|_2^2 \right)^{1/2}. \end{aligned}$$

Similar to (5.16), we obtain

$$(5.26) \quad k \sum_{i=1}^n \varepsilon_a^i(\mathbf{u}_h)(\mathbf{W}_i) \leq K \left(k^3 \sum_{i=1}^n \int_0^{t_i} e^{2\alpha s} (\|\mathbf{u}_h\|^2 + \|\mathbf{u}_{hs}\|^2) ds \right)^{1/2} \left(k \sum_{i=1}^n e^{-2\alpha t_i} \|\mathbf{W}_i\|_2^2 \right)^{1/2},$$

and

$$(5.27) \quad k \sum_{i=1}^n k(\partial_t \boldsymbol{\xi}_i, \partial_t \mathbf{W}_i) \leq k \left(k \sum_{i=1}^n e^{2\alpha t_i} \|\partial_t \boldsymbol{\xi}_i\|^2 \right)^{1/2} \left(k \sum_{i=1}^n e^{-2\alpha t_i} \|\partial_t \mathbf{W}_i\|^2 \right)^{1/2}.$$

Incorporating (5.25)-(5.27) in (5.24), and using Lemmas 3.2 and 5.3, we find that

$$(5.28) \quad k \sum_{i=1}^n e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 + \|\boldsymbol{\xi}_n\|_{-1}^2 \leq K k^2 e^{2\alpha t_n}.$$

□

Due to the non-smooth initial data, we need some intermediate results involving the “hat operator” which is defined as

$$(5.29) \quad \hat{\phi}_h^n := k \sum_{i=1}^n \phi_h^i.$$

This can be considered as discrete integral operator. We first observe, using (4.4), that

$$\begin{aligned} k \sum_{j=1}^i \beta(t_i - t_j) \phi_j &= \gamma e^{-\delta t_i} k \sum_{j=1}^i e^{\delta t_j} \phi_j \\ &= \gamma e^{-\delta t_i} \left\{ e^{\delta t_i} \hat{\phi}_i - k \sum_{j=1}^{i-1} \left(\frac{e^{\delta t_{j+1}} - e^{\delta t_j}}{k} \right) \hat{\phi}_j \right\} = \partial_t^i \left\{ k \sum_{j=1}^i \beta(t_i - t_j) \hat{\phi}_j \right\}. \end{aligned}$$

Here ∂_t^i means the difference formula with respect to i . Now rewrite the equations (5.7) (for $n = i$) as follows:

$$(5.30) \quad (\partial_t \boldsymbol{\xi}_i, \phi_h) + \mu a(\boldsymbol{\xi}_i, \phi_h) + \partial_t^i \left\{ k \sum_{j=1}^i \beta(t_i - t_j) a(\hat{\boldsymbol{\xi}}_j, \phi_h) \right\} = -E^i(\mathbf{u}_h)(\phi_h) - \varepsilon_a^i(\mathbf{u}_h)(\phi_h).$$

We multiply (5.30) by k and sum over 1 to n . Using the fact that $\partial_t \hat{\boldsymbol{\xi}}_n = \boldsymbol{\xi}_n$, we observe that

$$(5.31) \quad (\partial_t \hat{\boldsymbol{\xi}}_n, \phi_h) + \mu a(\hat{\boldsymbol{\xi}}_n, \phi_h) + a(q_r^n(\hat{\boldsymbol{\xi}}), \phi_h) = -k \sum_{i=1}^n (E^i(\mathbf{u}_h)(\phi_h) + \varepsilon_a^i(\mathbf{u}_h)(\phi_h)).$$

Lemma 5.5. *Under the assumptions of Lemma 5.2, the following estimate holds:*

$$(5.32) \quad \|\hat{\boldsymbol{\xi}}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\xi}}_i\|^2 \leq K k^2 (1 + \log \frac{1}{k}).$$

Proof. Choose $\phi_h = \hat{\boldsymbol{\xi}}_i$ in (5.31) for $n = i$, multiply by $k e^{2\alpha t_i}$ and then sum over $1 \leq i \leq n$. We drop the third term on the left hand-side of the resulting inequality due to non-negativity.

$$(5.33) \quad e^{2\alpha t_n} \|\hat{\boldsymbol{\xi}}_n\|^2 + \mu k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\xi}}_i\|^2 \leq k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i (|E^j(\mathbf{u}_h)(\hat{\boldsymbol{\xi}}_i)| + |\varepsilon_a^j(\mathbf{u}_h)(\hat{\boldsymbol{\xi}}_i)|).$$

From (5.2), we find that

$$k \sum_{j=1}^i |E^j(\mathbf{u}_h)(\hat{\boldsymbol{\xi}}_i)| \leq \frac{1}{2} \left(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} (s - t_{j-1}) \|\mathbf{u}_{hss}\|_{-1} ds \right) \|\nabla \hat{\boldsymbol{\xi}}_i\|.$$

Similar to the proof of Lemma 5.1, we split the sum in $j = 1$ and the rest to obtain

$$(5.34) \quad k \sum_{j=1}^i |E^j(\mathbf{u}_h)(\hat{\xi}_i)| \leq Kk(1 + \frac{1}{2} \log \frac{1}{k}) e^{-\alpha k} \|\nabla \hat{\xi}_i\|.$$

Therefore,

$$(5.35) \quad k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i |E^j(\mathbf{u}_h)(\hat{\xi}_i)| \leq \frac{\mu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_i\|^2 + Kk^2(1 + \log \frac{1}{k}) e^{2\alpha t_n}.$$

Similarly

$$(5.36) \quad k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i |\varepsilon_a^j(\mathbf{u}_h)(\hat{\xi}_i)| \leq \frac{\mu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_i\|^2 + Kk^2(1 + \log \frac{1}{k}) e^{2\alpha t_n}.$$

Incorporate (5.35)-(5.36) in (5.33) to complete the rest of the proof. \square

We are now in a position to estimate $L^\infty(\mathbf{L}^2)$ -norm of ξ_n .

Theorem 5.1. *Under the assumptions of Lemma 5.2, the following holds:*

$$(5.37) \quad t_n \|\xi_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 \leq Kk^2(1 + \log \frac{1}{k}),$$

where $\sigma_i = t_i e^{2\alpha t_i}$.

Proof. From (5.7) with $n = i$ and $\phi_h = \sigma_i \xi_i$, we obtain

$$(5.38) \quad \begin{aligned} \partial_t(\sigma_i \|\xi_i\|^2) - e^{2\alpha k} \left\{ \|\tilde{\xi}_{i-1}\|^2 + \left(\frac{1 - e^{-2\alpha k}}{k} \right) \sigma_{i-1} \|\xi_{i-1}\|^2 \right\} + 2\mu \sigma_i \|\nabla \xi_i\|^2 \\ + 2\sigma_i a(q_r^i(\xi), \xi_i) \leq -2E^i(\mathbf{u}_h)(\sigma_i \xi_i) - 2\varepsilon_a^i(\mathbf{u}_h)(\sigma_i \xi_i). \end{aligned}$$

We multiply (5.38) by k and sum it over $1 \leq i \leq n$ to find that

$$(5.39) \quad \begin{aligned} \sigma_n \|\xi_n\|^2 + (2\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1}) k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 \leq e^{2\alpha k} k \sum_{i=2}^{n-1} \|\tilde{\xi}_i\|^2 \\ - 2k \sum_{i=1}^n \sigma_i a(q_r^i(\xi), \xi_i) - 2k \sum_{i=1}^n E^i(\mathbf{u}_h)(\sigma_i \xi_i) - 2k \sum_{i=1}^n \varepsilon_a^i(\mathbf{u}_h)(\sigma_i \xi_i). \end{aligned}$$

As earlier, using (4.4), we note that

$$(5.40) \quad 2k \sum_{i=1}^n \sigma_i a(q_r^i(\xi), \xi_i) = 2k \sum_{i=1}^n \gamma a(\hat{\xi}_i, \sigma_i \xi_i) - 2k \sum_{i=2}^n k \sum_{j=1}^{i-1} \partial_t \beta(t_i - t_j) a(\hat{\xi}_j, \sigma_i \xi_i).$$

The first term can be handled as follows (for some $\varepsilon > 0$):

$$(5.41) \quad 2k \sum_{i=1}^n \gamma a(\hat{\xi}_i, \sigma_i \xi_i) \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 + K(\varepsilon, \mu, \gamma) k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_i\|^2.$$

For the second term, using similar technique as in (5.19), we observe that

$$(5.42) \quad \begin{aligned} 2k \sum_{i=2}^n k \sum_{j=1}^{i-1} \partial_t \beta(t_i - t_j) a(\hat{\xi}_j, \sigma_i \xi_i) \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 \\ + Kk \sum_{i=2}^n \left(k \sum_{j=1}^{i-1} e^{-\delta(t_i - t_j)} \left(\frac{e^{\delta k - 1}}{k} \right) e^{\alpha t_i} \|\nabla \hat{\xi}_j\| \right)^2 \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 + Kk \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_j\|^2. \end{aligned}$$

Combining (5.40)-(5.42), we find that

$$(5.43) \quad 2k \sum_{i=1}^n \sigma_i a(q_r^i(\xi), \xi_i) \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 + Kk \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_i\|^2.$$

From Lemma 5.1, we obtain for $r = 0$ and $s = 1$

$$(5.44) \quad 2k \sum_{i=1}^n \{E^i(\mathbf{u}_h)(\sigma_i \xi_i) + \varepsilon_a^i(\mathbf{u}_h)(\sigma_i \xi_i)\} \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 + Kk^2(1 + \log \frac{1}{k})e^{2\alpha t_n}.$$

Incorporate the estimates (5.43)-(5.44) in (5.39) and choose $\varepsilon = \mu/2$ to conclude

$$\sigma_n \|\xi_n\|^2 + (\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1})k \sum_{i=1}^n \sigma_i \|\nabla \xi_i\|^2 \leq Kk^2(1 + \log \frac{1}{k})e^{2\alpha t_n} + Kk \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\xi}_i\|^2.$$

We multiply by $e^{2\alpha t_i}$ and use Lemma 5.5 to complete the rest of the proof. \square

We now obtain estimates of $\boldsymbol{\eta}$ below. Hence forward, K_T means Ke^{KT} .

Lemma 5.6. *Assume (A1), (A2) and a spatial discretization scheme that satisfies conditions (B1), (B2) and (B4). Further, assume that \mathbf{U}^n and \mathbf{V}^n satisfy (4.6) and (5.5), respectively. Then, for some positive constant K , there holds*

$$(5.45) \quad \|\boldsymbol{\eta}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2 \leq K_{t_n} k(1 + \log \frac{1}{k}),$$

$$(5.46) \quad \|\boldsymbol{\eta}_n\|_1^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|_1^2 \leq K_{t_n}.$$

Proof. We shall only prove the first estimate as the second one will follow similarly. For $n = i$, we put $\phi_h = \boldsymbol{\eta}_i$ in (5.8), multiply by $ke^{2\alpha t_i}$ and sum over $1 \leq i \leq n \leq N$ to obtain as in (5.15)

$$(5.47) \quad \|\tilde{\boldsymbol{\eta}}_n\|^2 + 2\mu k \sum_{i=1}^n \|\nabla \tilde{\boldsymbol{\eta}}_i\|^2 \leq k \sum_{i=1}^{n-1} \frac{(e^{2\alpha k} - 1)}{k} \|\tilde{\boldsymbol{\eta}}_i\|^2 + 2k \sum_{i=1}^n e^{2\alpha t_i} \Lambda_h^i(\boldsymbol{\eta}_i).$$

We recall from (5.4) that

$$(5.48) \quad \Lambda_h^i(\boldsymbol{\eta}_i) = -b(\mathbf{u}_h^i, \xi_i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \xi_i, \boldsymbol{\eta}_i).$$

Using (3.8) and Lemma 5.2, we obtain the following estimates:

$$(5.49) \quad \begin{aligned} b(\mathbf{e}_i, \xi_i, \boldsymbol{\eta}_i) &\leq \|\xi_i\|^{1/2} \|\nabla \xi_i\|^{3/2} \|\nabla \boldsymbol{\eta}_i\| + \|\nabla \xi_i\| \|\boldsymbol{\eta}_i\| \|\nabla \boldsymbol{\eta}_i\| \\ &\leq \varepsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K \|\boldsymbol{\eta}_i\|^2 + Kk^{1/2}(1 + \log \frac{1}{k})^{1/2} \|\nabla \xi_i\|^2 \end{aligned}$$

$$(5.50) \quad b(\mathbf{u}_h^i, \xi_i, \boldsymbol{\eta}_i) \leq \varepsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K \|\nabla \xi_i\|^2$$

$$(5.51) \quad b(\mathbf{e}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) \leq \varepsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K(\|\nabla \xi_i\|^2 + \|\boldsymbol{\eta}_i\|^2).$$

Incorporate (5.49)-(5.51) in (5.48) and then in (5.47). Choose $\varepsilon = \mu/6$ and once again use Lemma 5.2. Finally, use discrete Gronwall's Lemma to complete the rest of the proof. \square

Remark 5.2. *Combining Lemmas 5.2 and 5.6, we note that*

$$(5.52) \quad \|\mathbf{e}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\mathbf{e}_i\|^2 \leq K_{t_n} k(1 + \log \frac{1}{k}),$$

$$(5.53) \quad \|\mathbf{e}_n\|_1^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\mathbf{e}_i\|_1^2 \leq K_{t_n}.$$

Therefore, we obtain suboptimal order of convergence for $\|\mathbf{e}_n\|$.

Below, we shall prove optimal estimate of $\|\mathbf{e}_n\|$ with help of a series of Lemmas.

Lemma 5.7. *Under the assumptions of Lemma 5.6, the following holds:*

$$(5.54) \quad \|\boldsymbol{\eta}_n\|_{-1}^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2 \leq K t_n k^2.$$

Proof. Put $\boldsymbol{\phi}_h = e^{2\alpha t_i} (-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i$ in (5.8) for $n = i$. Multiply the equation by $ke^{2\alpha i k}$ and sum over $1 \leq i \leq n \leq N$ to arrive at

$$(5.55) \quad \|\tilde{\boldsymbol{\eta}}_n\|_{-1}^2 + 2\mu k \sum_{i=1}^n \|\tilde{\boldsymbol{\eta}}_i\|^2 \leq \sum_{i=1}^{n-1} (e^{2\alpha k} - 1) \|\tilde{\boldsymbol{\eta}}_i\|_{-1}^2 + 2k \sum_{i=1}^n e^{2\alpha t_i} \Lambda_h^i (e^{2\alpha t_i} (-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i).$$

From (5.4), we find that

$$(5.56) \quad \begin{aligned} |2\Lambda_h^i ((-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i)| &\leq |2b(\mathbf{e}_i, \mathbf{u}_h^i, (-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i) \\ &\quad + b(\mathbf{u}_h^i, \mathbf{e}_i, (-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{e}_i, (-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i)|. \end{aligned}$$

For the first term on the right hand-side of (5.56), we use (3.8) to find that

$$(5.57) \quad |2b(\mathbf{e}_i, \mathbf{u}_h^i, -\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i)| \leq K \|\mathbf{e}_i\| \|\mathbf{u}_h^i\|_1 \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2}.$$

Also,

$$(5.58) \quad \begin{aligned} |2b(\mathbf{u}_h^i, \mathbf{e}_i, -\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i)| &\leq |(\mathbf{u}_h^i \cdot \nabla \mathbf{e}_i, -\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i)| + |(\mathbf{u}_h^i \cdot \nabla (-\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i), \mathbf{e}_i)| \\ &\leq |(\mathbf{u}_h^i \cdot \nabla \mathbf{e}_i, -\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i)| + K \|\mathbf{u}_h^i\|_1 \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2} \|\mathbf{e}_i\|. \end{aligned}$$

For $D_1 = \frac{\partial}{\partial x_1}$, we note that

$$(5.59) \quad \begin{aligned} (\mathbf{u}_h^i \cdot \nabla \mathbf{e}_i, -\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_i) &= \sum_{l,j=1}^2 \int_{\Omega} \mathbf{u}_{h,l}^i D_l(\mathbf{e}_{i,j}) (-\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_{i,j}) dx \\ &= - \sum_{l,j=1}^2 \int_{\Omega} \{D_l(\mathbf{u}_{h,l}^i) \mathbf{e}_{i,j} (-\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_{i,j}) + \mathbf{u}_{h,l}^i \mathbf{e}_{i,j} D_l((-\tilde{\Delta}_h^{-1} \boldsymbol{\eta}_{i,j}))\} dx. \\ &\leq K \|\mathbf{u}_h^i\|_1 \|\mathbf{e}_i\| \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2}. \end{aligned}$$

Finally, from (3.8), we find that

$$(5.60) \quad |2b(\mathbf{e}_i, \mathbf{e}_i, -\tilde{\Delta}_h^{-1} \mathbf{e}_i)| \leq K \|\mathbf{e}_i\| (\|\mathbf{e}_i\|_1 + \|\mathbf{e}_i\|^{1/2} \|\mathbf{e}_i\|_1^{1/2}) \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2}.$$

Now, combine (5.56)-(5.60) and use the fact that

$$\|\mathbf{e}_i\|_1 \leq \|\mathbf{u}_h^i\|_1 + \|\mathbf{U}^i\|_1 \leq K$$

to observe that

$$(5.61) \quad \begin{aligned} |2\Lambda_h^i ((-\tilde{\Delta}_h)^{-1} \boldsymbol{\eta}_i)| &\leq K \|\mathbf{e}_i\| \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2} \\ &\leq K \|\boldsymbol{\xi}_i\| \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{1/2} + K \|\boldsymbol{\eta}_i\|_{-1}^{1/2} \|\boldsymbol{\eta}_i\|^{3/2}. \end{aligned}$$

Incorporate (5.61) in (5.55) and use kickback argument to obtain

$$(5.62) \quad \|\tilde{\boldsymbol{\eta}}_n\|_{-1}^2 + \mu k \sum_{i=1}^n \|\tilde{\boldsymbol{\eta}}_i\|^2 \leq K k \sum_{i=1}^n \|\tilde{\boldsymbol{\eta}}_i\|_{-1}^2 + K k \sum_{i=1}^n \|\tilde{\boldsymbol{\xi}}\|^2.$$

Finally, use Lemma 5.4, apply discrete Gronwall's lemma and multiply the resulting estimate by $e^{-2\alpha t_i}$ to complete the rest of the proof. \square

Remark 5.3. From Lemmas 5.4 and 5.7, we have the following estimate

$$(5.63) \quad e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\mathbf{e}_i\|^2 \leq K_{t_n} k^2.$$

We need another estimate of $\boldsymbol{\eta}$ similar to the one in Lemma 5.5 and the proof will follow in a similar line. For that purpose, we multiply (5.8) by k and sum over 1 to n and similar to (5.31), we obtain

$$(5.64) \quad (\partial_t \hat{\boldsymbol{\eta}}_n, \phi_h) + \mu a(\hat{\boldsymbol{\eta}}_n, \phi_h) + a(q_r^n(\hat{\boldsymbol{\eta}}), \phi_h) = k \sum_{i=1}^n \Lambda_h^i(\phi_h).$$

Lemma 5.8. Under the assumptions of Lemma 5.6, the following holds:

$$(5.65) \quad \|\hat{\boldsymbol{\eta}}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2 \leq K_{t_n} k^2 (1 + \log \frac{1}{k}).$$

Proof. Choose $\phi_h = \hat{\boldsymbol{\eta}}_i$ in (5.64) for $n = i$, multiply by $k e^{2\alpha t_i}$ and then sum over $1 \leq i \leq n$ to observe as in (5.33)

$$(5.66) \quad e^{2\alpha t_n} \|\hat{\boldsymbol{\eta}}_n\|^2 + \mu k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2 \leq k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i |\Lambda_h^i(\hat{\boldsymbol{\eta}}_i)|.$$

We observe that

$$(5.67) \quad k \sum_{j=1}^i |\Lambda_h^j(\hat{\boldsymbol{\eta}}_i)| = k \sum_{j=1}^i \left| b(\mathbf{u}_h^j, \mathbf{e}_j, \hat{\boldsymbol{\eta}}_i) + b(\mathbf{e}_j, \mathbf{u}_h^j, \hat{\boldsymbol{\eta}}_i) + b(\mathbf{e}_j, \mathbf{e}_j, \hat{\boldsymbol{\eta}}_i) \right|.$$

Use (3.8), (5.52) and (5.53) to obtain

$$(5.68) \quad \begin{aligned} k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{i=1}^n |b(\mathbf{e}_j, \mathbf{e}_j, \hat{\boldsymbol{\eta}}_i)| &\leq K k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i \|\mathbf{e}_j\|^{1/2} \|\mathbf{e}_j\|^{3/2} \|\nabla \hat{\boldsymbol{\eta}}_i\| \\ &\leq K k \sum_{i=1}^n e^{2\alpha t_i} \left(k \sum_{j=1}^i \|\tilde{\mathbf{e}}_j\|^2 \right)^{1/4} \left(k \sum_{j=1}^i \|\tilde{\mathbf{e}}_j\|_1^2 \right)^{3/4} \|\nabla \hat{\boldsymbol{\eta}}_i\| \\ &\leq K_{t_n} k^2 (1 + \log \frac{1}{k}) + \varepsilon k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2. \end{aligned}$$

Similarly,

$$(5.69) \quad \begin{aligned} k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i |b(\mathbf{e}_j, \mathbf{u}_h^j, \hat{\boldsymbol{\eta}}_i)| &\leq K k \sum_{i=1}^n e^{2\alpha t_i} \left(k \sum_{j=1}^i \|\tilde{\mathbf{e}}_i\|^2 \right)^{1/2} \|\nabla \hat{\boldsymbol{\eta}}_i\| \\ &\leq K_{t_n} k^2 + \varepsilon k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2. \end{aligned}$$

and

$$(5.70) \quad \begin{aligned} k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i |b(\mathbf{u}_h^i, \mathbf{e}_i, \phi_h)| &\leq k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i \left(\frac{1}{2} |((\nabla \cdot \mathbf{u}_h^i) \mathbf{e}_i, \phi_h)| + |((\mathbf{u}_h^i \cdot \nabla) \phi_h, \mathbf{e}_i)| \right) \\ &\leq K k \sum_{i=1}^n e^{2\alpha t_i} \left(k \sum_{j=1}^i \|\tilde{\mathbf{e}}_i\|^2 \right)^{1/2} \|\nabla \hat{\boldsymbol{\eta}}_i\| \leq K_{t_n} k^2 + \varepsilon k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2. \end{aligned}$$

Combining these estimates, namely; (5.68)-(5.70) and putting $\varepsilon = \mu/6$, we conclude the rest of the proof. \square

We present below a Lemma with optimal estimate for $\boldsymbol{\eta}_n$.

Lemma 5.9. *Under the assumptions of Lemma 5.6, the following holds:*

$$(5.71) \quad t_n \|\boldsymbol{\eta}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n \sigma_i \|\boldsymbol{\eta}_i\|_1^2 \leq K_{t_n} k^2 (1 + \log \frac{1}{k}).$$

Proof. We choose $\phi_h = \sigma_i \boldsymbol{\eta}_i$ in (5.8) for $n = i$. Multiply the resulting equation by k and sum it over $1 < i < n$ to find that

$$(5.72) \quad \begin{aligned} \sigma_n \|\boldsymbol{\eta}_n\|^2 + 2\mu k \sum_{i=1}^n \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 &\leq K(\alpha) k \sum_{i=2}^{n-1} \|\tilde{\boldsymbol{\eta}}_i\|^2 - 2k \sum_{i=1}^n a(q_r^i(\boldsymbol{\eta}), \sigma_i \boldsymbol{\eta}_i) \\ &\quad + 2k \sum_{i=1}^n \Lambda_h^i(\sigma_i \boldsymbol{\eta}_i). \end{aligned}$$

As in (5.43), we obtain

$$(5.73) \quad 2k \sum_{i=1}^n \sigma_i a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 + Kk \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \hat{\boldsymbol{\eta}}_i\|^2.$$

Following the proof technique leading to the estimate (5.48), we observe that

$$(5.74) \quad 2k \sum_{i=1}^n \Lambda_h^i(\sigma_i \boldsymbol{\xi}_i) \leq \varepsilon k \sum_{i=1}^n \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 + Kk \sum_{i=1}^n \sigma_i (\|\nabla \boldsymbol{\xi}_i\|^2 + \|\boldsymbol{\eta}_i\|^2).$$

Substitute (5.73)-(5.74) in (5.72) and this completes the rest of the proof. \square

Theorem 5.2. *Under the assumptions of Lemma 5.6, following holds:*

$$(5.75) \quad \|\mathbf{e}_n\| \leq K_T t_n^{-1/2} k (1 + \log \frac{1}{k})^{1/2}.$$

Proof. Combine the Lemmas 5.1 and 5.9 to complete the rest of the proof. \square

Remark 5.4. *We need not split the error \mathbf{e} in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in order to obtain optimal error estimate (5.75). However for optimal error estimate in L^2 -norm which is uniform in time, we need to split the error $\mathbf{e}_n = \boldsymbol{\eta}_n - \boldsymbol{\xi}_n$.*

6 Uniform Error Estimate

In this section, we prove the estimate (5.75) to be uniform under the uniqueness condition $\mu - 2N\nu^{-1}\|\mathbf{f}\|_\infty > 0$, where N is given as in (3.17). We observe that the estimate (5.37) involving $\boldsymbol{\xi}_n$ is uniform in nature. Hence, we are left to deal with \mathbf{L}^2 estimate of $\boldsymbol{\eta}_n$.

Lemma 6.1. *Let the assumptions of Lemma 5.6 hold. Under the uniqueness condition $\mu - 2N\nu^{-1}\|\mathbf{f}\|_\infty > 0$ and under the assumption*

$$1 + (\frac{\mu\lambda_1}{2})k > e^{2\alpha k},$$

which holds for $0 < k < k_0$, $k_0 > 0$, the following uniform estimate hold:

$$(6.1) \quad \|\boldsymbol{\eta}_n\| \leq K \tau_n^{-1/2} k (1 + \log \frac{1}{k})^{1/2},$$

where $\tau_n = \min\{1, t_n\}$.

Proof. Choose $\phi_h = \boldsymbol{\eta}_i$ in (5.8) for $n = i$ to obtain

$$(6.2) \quad \partial_t \|\boldsymbol{\eta}_i\|^2 + 2\mu \|\nabla \boldsymbol{\eta}_i\|^2 + 2a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) \leq 2\Lambda_h^i(\boldsymbol{\eta}_i).$$

From (5.4), we find that

$$(6.3) \quad \Lambda_h^i(\boldsymbol{\eta}_i) = -b(\boldsymbol{\xi}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) - b(\boldsymbol{\eta}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i).$$

From the definition of N (see (3.17)), we note that

$$(6.4) \quad |b(\boldsymbol{\eta}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i)| \leq N \|\nabla \boldsymbol{\eta}_i\|^2 \|\nabla \mathbf{u}_h^i\|.$$

Again with the help of (3.8), we obtain

$$(6.5) \quad \begin{aligned} |b(\boldsymbol{\xi}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i)| + |b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i)| &\leq K \|\boldsymbol{\xi}_i\| \|\nabla \boldsymbol{\eta}_i\| \|\nabla \mathbf{u}_h^i\|^{1/2} \|\tilde{\Delta}_h \mathbf{u}_h^i\|^{1/2} \\ &\leq K \tau_i^{-3/4} k (1 + \log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\eta}_i\|. \end{aligned}$$

Since $\|\mathbf{e}_i\|_2 \leq \|\mathbf{u}_h^i\|_2 + \|\mathbf{U}^i\|_2 \leq K t_i^{-1/2}$, we conclude that

$$(6.6) \quad |b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i)| \leq K \tau_i^{-3/4} k (1 + \log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\eta}_i\|.$$

Therefore, from (6.4)-(6.6), we find that

$$(6.7) \quad |\Lambda_h^i(\boldsymbol{\eta}_i)| \leq N \|\nabla \boldsymbol{\eta}_i\|^2 \|\nabla \mathbf{u}_h^i\| + K \tau_i^{-3/4} k (1 + \log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\eta}_i\|.$$

We recall from [25] that

$$\limsup_{t \rightarrow \infty} \|\nabla \mathbf{u}_h(t)\| \leq \nu^{-1} \|\mathbf{f}\|_\infty,$$

and therefore, for large enough $i \in \mathbb{N}$, say $i > i_0$ we obtain from (6.7)

$$(6.8) \quad |2\Lambda_h^i(\mathbf{e}_i)| \leq 2N \nu^{-1} \|\mathbf{f}\|_\infty \|\nabla \boldsymbol{\eta}_i\|^2 + K \tau_i^{-3/4} k (1 + \log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\eta}_i\|.$$

With $\sigma_i = \tau_i e^{2\alpha t_i}$, we multiply (6.2) by $k\sigma_i$ and sum over $i_0 + 1$ to n to obtain

$$(6.9) \quad \begin{aligned} k \sum_{i=i_0+1}^n e^{2\alpha t_i} \{\partial_t \|\boldsymbol{\eta}_i\|^2 + 2\mu \|\nabla \boldsymbol{\eta}_i\|^2\} + 2k \sum_{i=i_0+1}^n e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) \\ \leq 2k \sum_{i=i_0+1}^n \sigma_i \Lambda_h^i(\boldsymbol{\eta}_i). \end{aligned}$$

Without loss of generality, we can assume that i_0 is big enough, so that, by definition $\tau_i = 1$ for $i \geq i_0$. We rewrite (6.9) as follows:

$$(6.10) \quad \begin{aligned} k \sum_{i=i_0+1}^n e^{2\alpha t_i} \{\partial_t \|\boldsymbol{\eta}_i\|^2 + 2\mu \|\nabla \boldsymbol{\eta}_i\|^2\} + 2k \sum_{i=1}^n e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) \\ \leq 2k \sum_{i=i_0+1}^n \sigma_i \Lambda_h^i(\boldsymbol{\eta}_i) + 2k \sum_{i=1}^{i_0} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i). \end{aligned}$$

We observe that the last term on the left hand-side of (6.10) is non-negative and hence is dropped.

$$(6.11) \quad \begin{aligned} e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 - \sum_{i=i_0+1}^{n-1} e^{2\alpha t_i} (e^{2\alpha k} - 1) \|\boldsymbol{\eta}_i\|^2 + \mu k \sum_{i=i_0+1}^n e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \\ + k \sum_{i=i_0+1}^n (\mu - 2N \nu^{-1} \|\mathbf{f}\|_\infty) e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \leq e^{2\alpha t_{i_0}} \|\boldsymbol{\eta}_{i_0}\|^2 + 2k \sum_{i=1}^{i_0} e^{2\alpha t_i} q_r^i(\|\nabla \boldsymbol{\eta}_i\|) \|\nabla \boldsymbol{\eta}_i\| \\ + K k^2 \sum_{i=i_0+1}^n \tau_i^{1/4} e^{2\alpha t_i} (1 + \log \frac{1}{k})^{3/4} \|\nabla \boldsymbol{\eta}_i\|. \end{aligned}$$

Under the assumption

$$1 + (\frac{\mu \lambda_1}{2}) k > e^{2\alpha k},$$

which holds for $0 < k < k_0$, $k_0 > 0$ with $0 \leq \alpha \leq \min \{ \delta, \frac{\mu \lambda_1}{2} \}$, we find that

$$(6.12) \quad \begin{aligned} & \frac{\mu}{2} k \sum_{i=i_0+1}^n e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 - \sum_{i=i_0+1}^{n-1} e^{2\alpha t_i} (e^{2\alpha k} - 1) \|\boldsymbol{\eta}_i\|^2 \\ & = k \sum_{i=i_0+1}^n \left(\frac{\mu}{2} - \frac{e^{2\alpha k} - 1}{k \lambda_1} \right) \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 \geq 0. \end{aligned}$$

Due to uniqueness condition, we arrive at the following:

$$(6.13) \quad k \sum_{i=i_0+1}^n (\mu - 2N\nu^{-1} \|\mathbf{f}\|_\infty) e^{2\alpha i} \|\nabla \boldsymbol{\eta}_i\|^2 \geq 0.$$

Following the proof techniques of (5.18)-(5.19), we obtain

$$(6.14) \quad 2k \sum_{i=1}^{i_0} e^{2\alpha t_i} q_r^i(\|\nabla \boldsymbol{\eta}_i\|) \|\nabla \boldsymbol{\eta}_i\| \leq Kk \sum_{i=1}^{i_0} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2.$$

And

$$(6.15) \quad \begin{aligned} & Kk^2 \sum_{i=i_0+1}^n \tau_i^{1/4} e^{2\alpha t_i} (1 + \log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\eta}_i\| \leq \frac{\mu}{4} k \sum_{i=i_0+1}^n \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 \\ & + Kk^2 (1 + \log \frac{1}{k}) k \sum_{i=i_0+1}^n e^{2\alpha t_i} \tau_i^{-1/2}. \end{aligned}$$

Incorporate (6.12)-(6.15) in (6.11), use Lemma 5.7 and (5.63) to observe that

$$(6.16) \quad \begin{aligned} & e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 + k \sum_{i=1}^n \sigma_i \|\nabla \boldsymbol{\eta}_i\|^2 \leq K_{t_0} k^2 + Kk^2 (1 + \log \frac{1}{k}) e^{2\alpha t_n} \\ & + Kk \sum_{i=1}^{i_0} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2. \end{aligned}$$

Multiply by $e^{-2\alpha t_i}$ and under the assumption that

$$(6.17) \quad k \sum_{i=1}^{t_0} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \leq K_{t_0} t_0^{-1} k^2 (1 + \log \frac{1}{k}).$$

we conclude that

$$\|\boldsymbol{\eta}_n\| \leq K t_n^{-1/2} k (1 + \log \frac{1}{k})^{1/2},$$

since $i_0 > 0$ is fixed. Combining this result with (5.37) we complete the rest of the proof. \square

We are now left with the proof (6.17).

Lemma 6.2. *Under the assumption of Lemma 5.6, the following holds*

$$k \sum_{i=1}^{i_0} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \leq K_{t_{i_0}} t_{i_0}^{-1} k^2 (1 + \log \frac{1}{k}).$$

Proof. In (5.47), we use

$$\begin{aligned} \Lambda_h^i(\boldsymbol{\eta}_i) &= -b_h(\mathbf{u}_h^i, \mathbf{e}_i, \boldsymbol{\eta}_i) - b_h(\mathbf{e}_i, \mathbf{U}^i, \boldsymbol{\eta}_i) \\ &\leq \frac{\mu}{4} \|\nabla \boldsymbol{\eta}_i\|^2 + K \|\mathbf{e}_i\|^2 (\|\tilde{\Delta}_h \mathbf{u}_h^i\| + \|\tilde{\Delta}_h \mathbf{U}^i\|), \end{aligned}$$

along with Lemma 5.7 and Theorem 5.2 to arrive at

$$(6.18) \quad \|\tilde{\boldsymbol{\eta}}_{i_0}\|^2 + \mu k \sum_{i=1}^{i_0} \|\nabla \tilde{\boldsymbol{\eta}}_i\|^2 \leq K_{t_{i_0}} k^2 + K_{t_{i_0}} t_{i_0}^{-1} k^2 (1 + \log \frac{1}{k}) k \sum_{i=1}^{i_0} e^{2\alpha t_i} t_i^{-1/2}.$$

This completes the rest of the proof. \square

7 Conclusion

In this paper, we have discussed optimal error estimates for the backward Euler method employed to the Oldroyd model with non-smooth initial data, the is, $\mathbf{u}_0 \in \mathbf{J}_1$. We have proved both optimal and uniform error estimate for the velocity. Uniform estimate is proved under uniqueness condition. The error analysis for the non-smooth initial data tells us that we need a few more proof techniques than the smooth case and proofs are more involved.

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